LOGARITHMS
TO 12 PLACES
AND THEIR USE IN
INTEREST CALCULATIONS

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INTEREST CALCULATIONS

BY CHARLES E. SPRAGUE

Author of "THE PHILOSOPHY OF ACCOUNTS"

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PREFACE.

The need of a logarithmic table for special cases, where the usual five-figure and seven-place results are insufficient, is often felt by the accountant and the actuary. Rough results will answer for approximativ purposes ; but where it is desirable, for instance, to construct a table of amortization, sinking fund or valuation of a lease at an unusual rate, for a large amount and for a great many years, exactness is desirable and becomes self-proving at the end.

It is, of course, a slower process than that for a few places, but as the figures from which all results are obtainable are containd in two pages instead of 200, there is, on the other hand, a great saving in the mechanical labor of turning leaves.

It also contains a thoro analysis of the entire doctrin of interest, explaining every process by the use of logarithms, as well as arithmetically and algebraically.

CHARLES E. SPRAGUE.

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PART I.

The Properties of Logarithms

PART I.

THE PROPERTIES OF LOGARITHMS.

1.—If we multiply 5 10's together, $10 \times 10 \times 10 \times 10 \times 10$, we may write the result as

$$100000$$
$$\text{or} \quad 10^5$$
$$\text{or} \quad \text{the fifth } \textit{power} \text{ of ten.}$$

The little "^{5}" is the *exponent* of the power. We may form a series of the powers of 10:

100000	or	10^5
10000		10^4
1000		10^3
100		10^2
10		10^1
1		10^0

2.—The following observations may then be made:

1. The number of the zeroes in the first colum is the exponent in the second.

2. Each term in the first colum is *one-tenth* of the one above it, while in the second colum each exponent is *one less* than the exponent above it. This leads to the result that $10^0 = 1$, which at first seems paradoxical.

3. If we multiply together any two terms in the first colum, we add the exponents in the second.

3.—**Logarithms** are auxiliary numbers having relation to a **base**. When the base is once fixt, every possible number has its logarithm. The customary and most convenient base is 10, because our whole system of numeration is based upon ten. The logarithms are simply exponents and we re-write the above series thus:

The base being 10,
100000 is the number whose logarithm is 5
or contracted,

100000.	*nl*	5
10000.	*nl*	4
1000.	*nl*	3
100.	*nl*	2
10.	*nl*	1
1.	*nl*	0
.1	*nl*	—1
.01	*nl*	—2
.001	*nl*	—3
.0001	*nl*	—4
.00001	*nl*	—5

The copula (*nl*) means "is the number whose logarithm is——;" while (*ln*) means "is the logarithm of the number——."

4.—We have here logarithms of a few numbers, but we need the logarithms of a great many others. All possible numbers must lie between some of the logarithms now ascertaind. The numbers between 1 and 10 must have their logarithms between 0 and 1; that is, the logarithms must be fractions, and these are exprest decimally to as many places as desired, the difficulty in calculation greatly increasing as the number of places is increast. Similarly, as the numbers of two figures lie between 10 and 100, their logarithms must lie between 1 and 2; that is, they must be 1 + a decimal fraction.

5.—We will now illustrate the properties of logarithms, confining our attention to the single-figure numbers 2, 3, 4, 5, 6, 7, 8 and 9, which are as follows, rounded at 12 places:

.301 029 995 664	*ln*	2
.477 121 254 720	*ln*	3
.602 059 991 328	*ln*	4
.698 970 004 336	*ln*	5
.778 151 250 384	*ln*	6
.845 098 040 014	*ln*	7
.903 089 986 992	*ln*	8
.954 242 509 439	*ln*	9

6.—The third observation in Art. 1 leads to the following rule :

The sum of the logarithms of several numbers is the logarithm of their product.

	2	*nl*	.301 029 995 664	
	3	*nl*	.477 121 254 720	
2 × 3 =	6	*nl*	.778 151 250 384	
	2	*nl*	.301 029 995 664	
	5	*nl*	.698 970 004 336	
2 × 5 =	10	*nl*	1.000 000 000 000	(See Art. 3)
	2	*nl*	.301 029 995 664	
2 × 10 =	20	*nl*	1.301 029 995 664	

7.—In the logarithms of 20, 200, 2000, 20,000, 200,000 2,000,000, etc., we shall find the same decimal part .301 029 995 664, (*ln* 2) preceded by the figures 1, 2, 3, 4, 5, 6, etc., indicating the distance from the units place of their left-hand figure, or the number of zeroes interpolated to hold that position. This is also true of any combination of figures; the decimal part of the logarithm is the same whatever their place-value, while the whole number prefixt indicates the place-value, being the number of places to the left of units.

Thus, if the logarithm of 2.378 is

.376 211 850 283, then

23.78	*nl*	1.376 211 850 283
237.8	*nl*	2.376 211 850 283
2,378	*nl*	3.376 211 850 283
23,780	*nl*	4.376 211 850 283
237,800	*nl*	5.376 211 850 283 ·
	etc.	etc.

8.—Where the number is less than unity (a decimal fraction) the characteristic or index (the prefixt figure) is negativ, altho the decimal (or mantissa) remains positiv. It is usual to put the minus sign over the characteristic:

.2378	(*nl*)	$\bar{1}$.376 211 850 283
.02378	(*nl*)	$\bar{2}$.376 211 850 283
.002378	(*nl*)	$\bar{3}$.376 211 850 283 ·
	etc.	etc.

Here the position of the left-hand figure of the number again determins the characteristic. $\bar{1}$ indicates that the left-hand figure, 2, is in the *first* place to the right of the unit place; $\bar{2}$ indicates that this figure is in the *second* place, and so on. The following list of characteristics will show that the left-hand figure of the combination, by its location to the right and left of the unit figure, determins the characteristic.

				Unit			
Places	0	000	000	00O.	000	000	000
Characteristics	9	8 7 6	5 4 3	2 1 0	$\bar{1}\,\bar{2}\,\bar{3}$	$\bar{4}\,\bar{5}\,\bar{6}$	$\bar{7}\,\bar{8}\,\bar{9}$

This principle saves a vast amount of time in the computation of logarithms, and also in their application.

9.—Since division is the converse of multiplication, it may be performd by subtraction as that is by addition.

The difference of the logarithms of two numbers is the logarithm of their quotient.

Required the quotient of 6 ÷ 2.

6	*nl*	.778 151 250 384	
2	"	.301 029 995 664	
6 / 2	*nl*	.477 121 254 720	*ln* 3

Required the value of ½ or 1 ÷ 2

1	*nl*	.000 000 000 000	
2	"	.301 029 995 664	
1 / 2	*nl*	$\bar{1}$.698 970 004 336	*ln* .5

Required the value of ⅓

1	*nl*	.000 000 000 000	
3	"	.477 121 254 720	
1 / 3	*nl*	$\bar{1}$.522 878 745 280	*ln* .3333.....

.522 878 745 280 is called the *cologarithm* of 3 or the logarithm of the reciprocal of 3.

10.—Powers of numbers are found by multiplication. Let it be required to find the third power of 2, which may be written 2^3 or $2 \times 2 \times 2$. By the process first shown

2	*nl*	.301 029 995 664	
2	"	.301 029 995 664	
2	"	.301 029 995 664	
$2 \times 2 \times 2 = 2^3$	*nl*	.903 089 986 992	*ln* 8

Required the square (2d power) of 3

$$\begin{array}{lll} 3 & nl & .477\ 121\ 254\ 720 \\ 3 & \text{``} & .477\ 121\ 254\ 720 \\ \hline 3^2 & nl & .954\ 242\ 509\ 440 \quad ln\ 9 \end{array}$$

In each of the above examples it would have been simpler to multiply the logarithm by the exponent.

2^3 *nl* (.301 029 995 664) $\times$ 3 = .903 089 986 992 *ln* 8.

3^2 *nl* (.477 121 254 720) $\times$ 2 = .954 242 509 440 *ln* 9.

Therefore, to "raise" a number to a certain power, we multiply its logarithm by the exponent and then find the number corresponding to the product-logarithm.

11.—The second power is usually called the *square*, and the third power the *cube*.

12.—If a certain number is a power of another, we call the latter a *root* of the former. Thus if $2^5 = 32$, we may say that the 5th root of 32 is 2. The usual way of expressing this is

$$\sqrt[5]{32} = 2, \text{ or}$$
$$32^{\frac{1}{5}} = 2.$$

Using the latter form gives a symmetrical list of exponents and their meanings:

a^n A positiv exponent denotes a power

a^{-n} A negativ exponent denotes the reciprocal of a power;

$a^{\frac{1}{n}}$ A fractional exponent denotes a root, or the root of a power;

a^1 The exponent ^{1} denotes the number itself;

a^0 The exponent ^{0} denotes unity.

13.—As roots are powers with fractional exponents, therefore roots are found (or *extracted*) by dividing logarithms insted of multiplying. Thus if it be required to find the 6th root of 64, we take (from Colum A of the Table of Factors) the logarithm of 64, and divide it by 6.

$64^{\frac{1}{6}}$ *nl* (1.806 179 973 984 /6) = .301 029 995 664 *ln* 2.

Therefore 2 is the 6th root of the number 64.

14.—Such an exponent as $\frac{3}{4}$ may require explanation. It signifies the third power of the fourth root or the fourth root of the third power.

15.—Fractional exponents may be represented as decimal, insted of vulgar fractions. Thus we may write $2^{.25}$ insted of $2^{\frac{1}{4}}$ or $3^{.5}$ for $3^{\frac{1}{2}}$. In fact, that is what most logarithms are: fractional exponents of 10, exprest decimally.

TABLES OF LOGARITHMS.

16.—The decimal fractions which constitute that part of the logarithm requiring tabulation are *interminate;* their values may be computed to any number of decimal places. If all the logarithms in a certain table are carried to 5 decimal places, it is called a 5-place table, and so on. Thus the logarithm of 2 has been computed, with great labor, to 20 places and even further.

2 *nl* .301 029 995 663 981 195 21+

In a 4-place table this would be rounded off to

.3010;

in a 7-place, .301 0300;

in a 10-place, .30102 99957;

in a 12-place, .301 029 995 664. The terminal decimal is never quite accurate, but is nearer than either the next greater or the next less.

17.—The number of figures in the numbers for which the logarithms are given must also be considered. The tables most in use, like those of Vega, Chambers and Babbage, are of five figures and seven places. A six-figure table would have to contain ten times as many logarithms and occupy ten times the space. A sixth and a seventh figure may be obtaind from them by interpolation. The United States Coast Survey tables (now out of print) are five-figure ten-places. Nine figures may be obtained by simple proportion, but the tenth is, for the most of the work, unreliable. Both of the foregoing systems give auxiliary tables of proportionate parts, or differences.

18.—PETER GRAY and ANTON STEINHAUSER have publisht tables of 24 and 20 places respectivly, but the plan for extending the numbers of figures is quite different from the simple interpolation above referd to. They both procede by subdividing the number into factors, and adding together the logarithms of those factors.

19.—All logarithmic calculations end with the ascertainment of a number which the problem calld for. The more

decimal places the tables give, the more exact the resulting number, or answer, will be, and the number of figures in the answer can never be *more* than the number of places in the final logarithm.

20.—I have selected twelve figures as the most useful limit for the accurate computation of interest problems, that being the kind for which the work is specially designd. The logarithms are given to two figures and thirteen places, the extra place insuring the accuracy of the 12th, which would otherwise sometimes be 1, 2 or even 3 units in error, thru the roundings being preponderant in one direction or the other.

21.—The method used is that of factoring, it being possible to construct the logarithm of any number of twelve figures or less (900,000,000,000 in all) by some combination of the 584 logarithms given on the two pages of the Table of Factors.

Colum A contains numbers of two figures, 11 to 99, and their logarithms to thirteen places.

Colum B contains the logarithms of four-figure numbers 1.001 to 1.099, each beginning with 1.0..

Colum C contains the logarithms of six-figure numbers 1.00001 to 1.00099, each beginning with 1.000...

Colum D, 1.0000001 to 1.0000099, beginning with one and five zeroes.

Colum E, 1.000000001 to 1.000000099, beginning with one and seven zeroes.

Colum F, 1.00000000001 to 1.00000000099, beginning with one and nine zeroes.

For example, opposit 34 in the table we find :

A	.531 478 917 042,3	*ln*	3.4
B	.014 520 538 757,9	*ln*	1.034
C	.000 147 635 027,3	*ln*	1.00034
D	.000 001 476 598,7	*ln*	1.0000034
E	.000 000 014 766,0	*ln*	1.000000034
F	.000 000 000 147,7	*ln*	1.00000000034

By omitting all the prefixt zeroes, the printed table is made very compact, each line containing only 53 figures insted of 78. It will be understood hereafter that C 34, for example, means the number 1.00034, and F 34 means 1.00000000034.

TO FIND THE NUMBER WHEN THE LOGARITHM IS GIVEN.

22.—In this process there are two stages: first, to divide the logarithm into a number of partial logarithms among those containd in the T F (Table of Factors); second, to multiply together the numbers corresponding to these logarithms. Of course the decimal part only of the logarithm is used and the number has the position of its units figure determind from the characteristic.

23.—Let the logarithm .753 797 472 366,5 be one which has been obtaind as the result of an operation, and the corresponding number be required. Search in Colum A for the highest logarithm which does not exceed the given one. This is found to be .748 188 027 006,2, which stands opposit 56.

Subtracting from	.753 797 472 366,5
A 56	.748 188 027 006,2
we have the remainder...............	. 5 609 445 360,3
This is smaller than any logarithm in Colum A. We search for it in Colum B and find opposit 13 precisely the same figures................	5 609 445 360,3

These two logarithms added together make the given logarithm; hence the product of their numbers gives the number required.

To multiply 56 by 1.013

$$\left\{\begin{array}{r} 56 \\ 1013 \\ \hline 56 \\ 56 \\ 168 \\ \hline 56728 \end{array}\right\} \text{ or } \begin{array}{lr} & 1013 \\ \hline 5 & 5065 \\ 6 & 6078 \\ \hline & 56728 \end{array}$$

24.—This process may be greatly simplified as follows, placing the figures of the multiplier in vertical order at the side:

$$\begin{array}{r|r} & 56 \\ 1 & 56 \\ 3 & 168 \\ \hline & 56728 \end{array} \quad \text{or} \quad \begin{array}{rr} & 56 \\ 13 \times 5 & 065 \\ 13 \times 6 & 078 \\ \hline & 56728 \end{array}$$

Notice that the first product is moved *two* colums to the right of the multiplicand.

25.—We will now take a little larger logarithm.....		753 911 659 107,4
and continue the subtraction	A 56	748 188 027 006,2
		5 723 632 101,2
	B 13	5 609 445 360,3
		114 186 740,9
	C 26	112 901 888,7
		1 284 852,2
	D 29	1 259 452,2
		25 400,0
	E 58	25 189,1
		210,9
	F 48	208,5
		2,4
	G 55 +	2,4

There is no colum G; but it is found by simply taking the first two figures from E. It may be either 55 or 56, which may make the thirteenth figure of the result doutful, but probably not the twelfth.

		5600
	1	56
	3	168
		567280000
See Note 1.	2	113456
	6	340368
		5674274928000
See Note 2.		★
	2	11.348550 ||
	9,	5106847
		5674291383397
See Note 3.	5	• 2 8.3.7.1.5.
	8	• 45394
	4	• 2270
	8	• 454
	5	• 28
	5	3
See Note 4.		5674291715 26,

Note 1.—The second multiplication jumps its right-hand figure (6) *four* places to the right, which may be markt off by four zeroes, or four dots.

Note 2.—Having extended the product to include the 13th figure, contraction begins in this multiplicand; its first figure used being the 7th (markt ★) allowing for the carrying from the 8th. Thus the starting point for this multiplication is moved *six* places *back*.

Note 3.—The multiplicand need no longer be extended, as has been done at successiv stages above, but remains the same to the end. For convenience, dots may be placed in advance under the first figure to be used in multiplication in each line.

Note 4.—The thirteenth figures are added, but only used for carrying to the twelfth. In this example the total of the last colum is 31, but it does not appear, except as contributing 3 to the next colum.

The dot below a figure indicates where the contracted multiplication begins, all the figures to the right being ignored, except as to their carrying power.

25.—Another example in which there is no suitable logarithm in A and we must begin with B.

Required the number for log. 011 253 170 227

FORMATION OF NUMBER FROM LOGARITHM.

Logarithm	0	1	1	2	5	3	1	7	0	1	2	7	0	
A —														
B 26		1	1	1	4	7	3	6	0	7	7	5	8	
				1	0	5	8	0	9	3	5	1	2	
C 24				1	0	4	2	1	8	1	7	0	0	
						1	5	9	1	1	8	1	2	
D 36						1	5	6	3	4	5	7	3	
								2	7	7	2	3	9	
E 63								2	7	3	6	0	5	6
										3	6	3	4	3
F 83										3	6	0	5	
												2	9	8
G 67 65+												2	9	
A —														
B	1	0	2	6										
26	1	0	2	6				★						
C 2					2	0	5	2						
4						4	1	0	4					
	1	0	2	6	2	4	6	2	4					
							•							
D 3							3	0	7	8	7	3	9	
6						•		6	1	5	7	4	8	
	1	0	2	6	2	4	9	9	3	4	4	8	7	
E 6					•				6	1	5	7	5	
3				•						3	0	7	9	
F 8			•								8	2	1	
3		•										3	1	
G 6	•												6	
7													1	
	1	0	2	6	2	5	0	0	0	0	0	0		

In this example we illustrate the procedure when B furnishes the first logarithm. It also shows the convenience of using paper ruled for the purpose.

26.—In order to set down the partial products without hesitation, remember the numbers 2, 4, 6.

In multiplying by B
  the first figure of the product moves *two* places to the right.

In multiplying by C
  the first figure of the product moves *four* places to the right.

In multiplying by D
  the first figure of the *multiplicand* moves *six* places to the *left*.

27.—The following rule may now be formulated for this process:

Rule.—1. By successiv subtractions separate the given logarithm into a series of partial logarithms found in the colums of the T F, setting opposit each its letter and number.

2. By successiv multiplications find the product of all the numbers thus found, allowing, in the placing of the partial products, for the prefixt 1 and zeroes.

28. The work may be made to occupy fewer lines by setting down the factors E, F and G as one number at the top, multiplying it by A and incorporating it thereafter as one multiplicand with the preceding figures. The result will not be affected. Let the factors be, as before: A 56 B 13 C 26 D 29 E 58 F 48 G 55.

```
                          E F G
                        5 8 4 8 5 5
A 56                  -------------
                      2 9 2 4 2 7 5
                        3 5 0 9 1 3
        -------------------------
        5 6 0 0 0 0 0 3 2 7 5 1 9
B 13        5 6 0 0 0 0 0 3 2 7 5
            1 6 8 0 0 0 0 0 9 8 2
        -------------------------
        5 6 7 2 8 0 0 3 3 1 7 7 6
C 26          1 1 3 4 5 6 0 0 6 6
                3 4 0 3 6 8 0 2 9
        -------------------------
        5 6 7 4 2 7 5 2 5 9 8 7 1
D 29                1 1 3 4 8 5 5 1
                      5 1 0 6 8 4 8
        -------------------------
        5 6 7 4 2 9 1 7 1 5 2 6 0
```

29.—Required the number whose logarithm is .5 or ½.

	.500 000 000 000,0
A 31	491 361 693 834,3
	8 638 306 165,7
B 20	8 600 171 761,9
	38 134 403,8
C 08	34 742 168,9
	3 392 234,9
D 78	3 387 483,7
	4 751,2
E 10	4 342,9
	408,3
F 94	408,2
G 03	0,1

The resulting factors

A 31 B 20 C 08 D 78 E 10 F 94 G 03

when combined produce the result 3.16227766017.

30.—The multiplication illustrates how zeroes are treated when they occur in the multipliers.

31.—The result is the square root of 10, to 12 places, as may be demonstrated by multiplying 3.16227766017 by itself.

Method by Multiples.

32.—In order to facilitate the multiplication of the factors, A, B, C, etc., Mr. A. S. Little, of St. Louis, has devised a Table of Multiples, giving the product of each number from 1 to 9 by every number from 2 to 99. (See page 35.) Thus the multiples of 89 read in one line as follows:

1	2	3	4	5	6	7	8	9
089	178	267	356	445	534	623	712	801

Then, if it be desired, for example, to multiply 68792341 by 89, we would select from the above table

under 6	534
8	712
7	623
9	801
2	178
3	267
4	356
1	089
	6122518349

We have thus multiplied each figure of the multiplicand by both figures of the multiplier, setting down each partial product unhesitatingly.

33.—The work may be made more compact by piling the partial products like bricks, using only three lines:

```
534.801,356,
  712,178,089
    623,267,
-------------
6122518349
```

34.—Three figures must be set down for each partial product, even if the first be a zero.

35.—To use this method in combining the factors of a number, the letters A, B, C, etc., are written above alternate figure spaces, which is facilitated by the use of paper properly ruled. Then the first partial product under each letter is placed with its middle figure under that letter at the top.

36.—The following is an example of a combination already performd in another form :

```
         A  B  C  D  E  F  G
A 56     1           584855
                     ------
                    280,448
                      448,280
                        224,28
         ---------------------
         56         327519
B 13       065        039,06
              078       026,2
                          091
         ---------------------
         5672800331778
C 26           130.052,000,
                 156,208,078
                   182.000,08
         ---------------------
         5674275259864
D 29               145.116,15
                     1740581
                       203.203
         ---------------------
         5674291715 26
```

37.—Mr. Little has also suggested a process for verifying a numerical result by using a different set of factors in a second operation.

38.—Required the number corresponding to

.305 773 384 163,0

The factors are A 20 B 10 C 97 D 21 E 94 F 94 G 33. The number is 2.02195383809.

In order to check the result and make sure of perfect accuracy, we may solve the problem a second time, using two subtrahends from A. The first subtraction may be of any suitable number; 11 is found to give the greatest facility.

	305 773 384 163,0
A 11	041 392 685 158,2
	264 380 699 004,8
A 18	255 272 505 103,3
	9 108 193 901,5
B 21	9 025 742 086,9
	82 451 814,6
C 18	78 165 972,0
	4 285 842,6
D 98	4 256 065,1
	29 777,5
E 68	29 532,0
	245,5
F 56	243,2
	2,3
G 53	2,3

39.—The remainder of the operation may be by either method:

```
      A B C D E F G
A 18              6 8 5 6 5 3
                  5 4 8 5 2 2
      1 8       1 2 3 4 1 7 5
A 11    1 8       1 2 3 4 1 8
      1 9 8     1 3 5 7 5 9
B 21    3 9 6       2 7 1 5
          1 9 8       1 3 6
      2 0 2 1 5 8 0 1 3 8 6 1 0
C 18          2 0 2 1 5 8 0 1 4
              1 6 1 7 2 6 4 1 1
      2 0 2 1 9 4 4 0 2 3 0 3 5
D 98            1 8 1 9 7 4 9 6
                  1 6 1 7 5 5 5
      2 0 2 1 9 6 3 8 3 8 0 9
```

```
      A B C D E F G
A 18              6 8 5 6 5 3
                  5 4 8 5 2 2
      1 8       1 2 3 4 1 7 5
A 11    1 8       1 2 3 4 1 8
      1 9 8     1 3 5 7 5 9
B 21    0 2 1       0 2 1,1 5
          1 8 9       0 6 3,1
            1 6 8       1 0 5
      2 0 2 1 5 8 0 1 3 8 6 1 0
C 18      0 3 6,0 1 8,0 0 0,1
            0 0 0,0 9 0,0 1 8
              0 3 6,1 4 4,0 5
      2 0 2 1 9 4 4 0 2 3 0 3 4
D 98          1 9 6,0 9 8,3 9
                0 0 0,8 8 2,0
                  1 9 6 3 9 2
      2 0 2 1 9 6 3 8 3 8 0 9
```

In this example the first method appears to be preferable, especially in the earlier part.

TO FORM THE LOGARITHM OF A NUMBER.

40.—This consists in two processes: first, the number is separated into a series of factors corresponding to the six colums of the thirteen-place table; second, the logarithms of these factors are copied from the table and added together.

41.—The factoring is effected by a progressiv division, the divisor receiving successivly more and more of the figures of the number.

42.—To illustrate this division we will assume a number in which the division will be soon completed.

To find the logarithmic factors, A, B, C, etc., of 5.6728. First extend the number to 12 places, 567 280 000 000. The first factor A is always the first two figures of the number itself.

```
A 56)56 7 2 80 000 000 (1.013 B
     56
       7 2
       5 6
       -----
       1 6 8
       1 6 8
       -----
```

It will readily be seen that one 56 might have been omitted.

```
A 56)7 280 000 000 (B 13
     5 6
     1 68
     1 68
     ----
```

Turning then to the Table we have only to set down the logarithms of these two factors:

A 56	*nl*	748 188 027 006,2
B 13	*nl*	5 609 445 360,3
56728	*nl*	753 797 472 366 5

B 13 may be regarded as an abbreviation of 1.013.

43.—We will now give an example where a second divisor, at least, is required.

```
          A 56) 7 4 2 9 1 7 1 5 2 6 (B 13
                5 6
                -----
                1 8 2
                1 6 8
                -----
A B 56 728 )    1 4
```

The second divisor is the product of A and B. It might be obtaind in either of three ways.

```
By multiplication      56 × 1.013 = 56728
By addition            56
                    +    56
                    +   168
                       -----
                       56728
But the easiest way is
by subtraction         56742 five figures of the number
                    —     14 the remainder
                       -----
                       56728
```

This is the proper method for forming all divisors after the first; subtract the remainder from the original number so far as used.

44.—We resume the division, bringing down *four* more figures, to the ninth inclusiv.

```
   A B ) 5 6 7 2 8       ) 1 4 9 1 7 1 5 2 6 (C 26
                           1 1 3 4 5 6
                           -----------
                             3 5 7 1 5 5
                             3 4 0 3 6 8
                           -------------
A B C ) 5 6 7 4 2 7,5 )     *1 6 7 8 7 2 6 (D 29
                               1 1 3 4 8 5 5
                               -------------
                                 5 4 3 8 7 1
                                 5 1 0 6 8 5
                                 -----------
        5 6 7 4 2 9,1 7+           3 3 1 8 6 (E 58
                                   2 8 3 7 1
                                   ---------
                                     4 8 1 5
                                     4 5 3 9
                                     -------
                                       2 7 6 (F 48,7
                                       2 2 7
                                       -----
                                         4 9
                                         4 5
                                         ---
                                           4
```

The third divisor A B C is also formd by subtracting from

```
the number                  5 6 7 4 2 9 1 7 1 5
          * the remainder             1 6 7 8 7
                            -------------------
                            5 6 7 4 2 7 4 9 2 8
```

As only six figures are needed for the divisor and one for carrying, this is rounded up to 5 6 7 4 2 7,5

The fourth divisor is practically the number itself so far as needed, and this lasts to the end.

45.—The entire process is now repeated, but for greater accuracy in the twelfth figure we will divide out to the thirteenth.

```
                         A 56) 7 4 2 9 1 7 1 5 2 6,0 (B 13
                               5 6
                               ---
                               1 8 2
                               1 6 8
                               -----
               A B 56 728)       1 4 9 1 7 1          (C 26
                                 1 1 3 4 5 6
                                 -----------
                                   3 5 7 1 5 5
                                   3 4 0 3 6 8
                                   -----------
        A B C 56 742 749 28)         1 6 7 8 7 2 6,0 (D 29
[Contracted division begins here]    1 1 3 4 8 5 5,0
                                     ---------------
                                       5 4 3 8 7 1,0
                                       5 1 0 6 8 4,7
                                       -------------
               56 742 92)                3 3 1 8 6,3 (E 58
                                         2 8 3 7 1,4
                                         -----------
                                           4 8 1 4,9
                                           4 5 3 9,4
                                           ---------
                                             2 7 5,5 (F 48
                                             2 2 7,0
                                             -------
                                               4 8 5
                                               4 5 4 (G 55
                                               -----
                                                 3 1
                                                 2 8
                                                 ---
                                                   3
```

It remains only to add together the logarithms:

A 56	(nl)	748 188 027 006,2
B 13	"	5 609 445 360,3
C 26	"	112 901 888,7
D 29	"	1 259 452,2
E 58	"	25 189,1
F 48	"	208,5
G 55	"	2,4
567 429 171 526	(nl)	753 911 659,107

46.—The figures in the last colum are only used for carrying to the twelfth, which otherwise would give 8 insted of 7.

47.—We may now formulate the following rule for finding the logarithm:

Rule.—1. Make the number to 13 figures, by adding cifers or cutting off decimals.

2. Cut off the two left-hand figures by a curve, giving A.

3. Divide the next three figures by A, giving the two figures of B, and a remainder.

4. Form the second divisor A B, by subtracting the remainder from the first five figures of the number.

5. Bring down four more figures to the remainder and divide by A B, giving the two figures of C and a remainder.

6. Form the third (and last) divisor A B C by subtracting the remainder from ten figures of the number.

7. Divide the remaining figures by the third divisor. As there are ten figures in the divisor and only eight in the dividend, contraction begins immediately. Having obtaind the figures of D, the divisor for E, F and G is simply the number itself contracted.

8. Write down the logarithms of A, B, C, D, E and F, obtaind from the several colums of T F; also that of G, being the first two figures of E. The sum will be the logarithm, the thirteenth figure being used for carrying only.

48.—It is advisable to make all logarithmic computations on paper ruled with thirteen down-lines, every third being darker. A specimen is given on page 36.

49. A few examples for practis are given below with the factors and the solution:

5674 = A 56 B 13 C 21 D 15 E 35 F 42 G 70
log. 5674 = 3.753 889 331 458

38.8586468578 = A 38 B 22 C 58 D 31 E 39 F 02 G 25
log. do. = 1.589 487 673 453

3.1415926535898 + = A 31 B 13 C 41 D 16 E 33 F 11 G 91
log. do. = .497 149 872 694

(This number is the ratio of the circumference of a circle to its diameter.)

1.02625 = B 26 C 24 D 36 E 63 F 83
log. do. = .011 253 170 127

This number begins with an expression of the form B (1.026), hence no division by A occurs. 1026 is the first divisor.

B 1026) 2 5 0 0 C 24
2 0 5 2
4 4 8 0
4 1 0 4
B C 102624624) 3 7 6 0 0 0,0 D 36
3 0 7 8 7 3,9
6 8 1 2 6,1
6 1 5 7 4,8
6 5 5 1,3 E 63
102625) 6 1 5 7,5
3 9 3,8
3 0 7,9
8 5,9 F 83
8 2,1
3,8
3,1
7 G 70

B 26	011 147 360 775,8
C 24	104 218 170,0
D 36	1 563 457,3
E 63	27 360,6
F 83	360,5
G 70	3,0
	011 253 170 127

This result will be found also in the Table of Interest-Ratios, but even more extended.

Logarithms to Less than 12 Places.

50.—The T F may be cut down to any lower number of places. In the example in Art. 45 it may be required to give 9 places only, the tenth being used for carrying. We cut down the original logarithm to ten figures, with a comma after the ninth and it becomes

	753 911 659,1
A 56	748 188 027,0
	5 723 632,1
B 13	5 609 445,4
	114 186,7
C 26	112 901,9
	1 284,8
D 29	1 259,5
	25,3
E 58	25,2
F 24	1

A	5 6
B 1	5 6
3	1 6 8
	5 6 7 2 8 0 0 0 0
C 2	1 1 3 4 5 6
6	3 4 0 3 6,8
	5 6 7 4 2 7 4 9 2,8
D 2	1 1 3 4,9
9	5 1 0,7
E 5	2 8,4
8	4,5
F 2	1
	5 6 7 4 2 9 1 7 1,4

The number is slightly in error in its tenth place, but correct to the ninth.

51.—If a table of factors for 18 or some other number of places should hereafter be prepared, the methods which have been explaind would be applicable.

MULTIPLYING UP.

52.—Mr. Edward S. Thomas, of Cincinnati, has suggested another method for obtaining the factors of the number in forming its logarithm.

53.—It procedes by multiplication insted of division, the latter operation being notably the more laborious. The number, at first taken as a decimal less than 1, is successivly multiplied *up* to produce 1.000,000,000,0 and these multipliers are the A, B, C, D, E, F and G, whose logarithms added together make the cologarithm, from which the logarithm is easily obtaind.

54.—A is a number of two figures, a little less than the reciprocal of the number, which will be calld the sub-reciprocal of its two initial figures. A Table of Sub-Reciprocals is given on page 33. The number multiplied by A will always give a product beginning with 9. B is always the arithmetical complement of the two figures following the nine, or the remainder obtaind by subtracting those two figures from 99. Multiplication by B will usually give a result beginning with 999. C is the next complement and gives 5 9's, 999,99. D similarly brings 999,999,9**,***,*. No further multiplication is necessary, when D has been used; the six figures in the places of the stars are the complements of E, F and G.

55.—To illustrate, let it be required to obtain the logarithm to the 12th place of 3.14 159 265 359 0. The object is to multiply .314 159 265 359 up to 1.000 000 000 000 0. The first step is to find the sub-reciprocal of 31, or A. Turning to the Table of Sub-reciprocals, opposit 31 we find 31, by which we multiply.

		.3141592653590	
	A 31	.9424777960770	
		314159265359	
		.9738937226129	One 9 has been secured
99 — 73 = 26 B 26 is therefore the next multiplier; dropping the last two figures		194778744523	
		58433623357	
		.9992149594009	Three nines secured
(99—21)	C 78	6994504716	
		799371968	
		.999943470693	Five nines
(99—43)	D 56	49999718	
		5999966	
		.999999470377	Seven nines
(99—47)	E 52	52	
(99—03)	F 96	96	
(100—77)	G 23	23	

A 31 *nl*	.4913616938343
B 26	111473607758
C 78	3386176522
D 56	24320423
E 52	225833
F 96	4169
G 23	10
colog.	0.502850127306
log.	$\bar{1}$.497149872694

56.—It may happen, in the course of multiplication, that the complement of the figures following the 9 does not suffice to secure two nines more. In this case, another supplementary multiplication must take place. This occurs in the following example, which has alredy been solvd.

57.—Required the logarithm of the number
567 429 171 526.

In this example the C multiplication also requires an additional figure. This seldom occurs.

	.567 429 171 526 0
A 17	.397 200 420 068 2
	.964 629 591 594 2
B 35	28 938 887 747 8
	4 823 147 958 0
	.998 391 627 300 0
" 01	998 391 627 3
	.999 390 018 927 3
C 60	599 634 011 4
	.999 989 652 938 7
" 01	9 999 896 5
	.999 999 652 835 2
D 03	299 999 9
	.999 999 952 835 1
	47 164 9

A 17	230 448 921 378 3
B 35	14 940 349 792 9
01	434 077 479 3
C 60	260 498 547 4
01	4 342 923 1
D 03	130 288 3
E 47	20 411 8
F 16	69 5
G 49	2 1
	.246 088 340 892 7
	.753 911 659 107 3

As the multiplication by 35 brings only 998 insted of 999, we multiply again by B 01, which brings it up.

58.—In the next example there is a large defect in B, which requires an additional multiplication by 7.

	110 175	
A 83	881 400	(83, subreciprocal of 11)
	33 052 5	
	914 452 5	
B 85	73 156 200	
	4 572 262 5	
	992 180 962 5	
B 07	6 945 266 737,5	
	999 126 229 237,5	
C 87	799 300 983,4	
	69 938 836,0	
	999 995 469 056,9	
D 45	3 999 981,9	
	499 997,7	
	999 999 969 036,5	
	30 963,5	

A 83	919 078 092 376,1
B 85	35 429 738 184,5
B 07	3 029 470 553,6
C 87	377 670 935,8
D 45	1 954 320,8
E 30	13 028,8
F 96	416,9
G 35	1,5
	957 916 940 818 0
	042 083 059 182 0

The number 11075 was purposely selected, very slightly in excess of the highest number in colum B, so as to produce the shortage of 7.

59.—Little's Table of Multipliers may be used in the multiplication, as in the following example. It will be found that the logarithm when computed has the same figures as the number itself; a remarkable peculiarity which no other combination of figures can possess.

		. 137 128 857 423 9
A	71	0 710 715 682 846 4
		213 142 355 142 0
		49 756 849 721 3
		. 973 614 887 709 7
B	26	23 415 620 818 2
		1 820 262 080 0
		078 104 182 2
		. 998 928 874 790 1
"	01	998 928 874 8
		. 999 927 803 664 9
C	07	69 994 946 3
		. 999 997 798 611 2
D	22	1 981 981 8
		198 198 0
		19 815 4
		. 999 999 998 606 4
		01 393 6
		E F G

A	71	851 258 348 719 1
B	26	11 147 360 775 8
B	01	434 077 479 3
C	07	30 399 549 8
D	22	955 446 8
E	01	434 3
F	39	169 4
G	36	1 6
		862 871 142 576 1
		.137 128 857 423 9

which is the log. of 1.371 288 574 239

SIGNS OF THE CHARACTERISTIC.

60.—We have seen (Art. 8) that while the decimal part of the logarithm is always positiv, the characteristic is often negativ and has the minus sign above it.

61.—In adding together several logarithms with different signs, the positivs and the negativs must be added separately; the less sum must be subtracted from the greater, and the remainder has the sign of the greater sum. The carrying from the decimal part counts with the positivs.

34	*nl.*		1.531 478 917 042,3
2900	"		3.462 397 997 899,0
.73	"		$\bar{1}$.863 322 860 120,5
.056	"		$\bar{2}$.748 188 027 006,2
The sum of the decimals is....			2.605 387 802 068,0
The positivs are			1.
and			3
Total		+ 6	
The negativs are	$\bar{1}$		
	$\bar{2}$	— 3	
Sum of the logarithms		+	3.605 387 802 068,0

The decimal point in the result must follow the fourth figure, as indicated by the characteristic 3.

62.—In subtracting one logarithm from another, when the decimal of the subtrahend is the greater, and a unit is "borrowed," the unit is considered as one more negativ: ~~but~~ the total characteristic changes its signs from plus to minus or from minus to plus.

in the result therefore does not change si

290		2 462 397 997 899,0
.0058		$\bar{3}$.763 427 993 562,9
Operate on the decimals only........		0.462 397 997 899,0
		0.763 427 993 562,9
		$\bar{1}$.698 970 004 336,1
Negativ from subtrahend	$\bar{3}$	
Total negativ.....	$\bar{4}$	
Sign changed.....	+ 4	
From minuend ...	+ 2	
		6.698 970 004 336,1 *ln* 5 000 000

wrong.

63.—To multiply a logarithm having a negativ characteristic (in order to obtain a power of a decimal), multiply the decimal part and the characteristic separately and add the two together:

	$\bar{2}.301\ 029\ 995\ 664,0$	$\times\ 5$
Decimal part.......	$1.505\ 149\ 978\ 320,0$	
Characteristic	$\bar{1}\bar{0}.$	
	$\tilde{9}.505\ 149\ 978\ 320,0$	

Therefore the 5th power of .02 is .000 000 003 2.

64.—To divide a logarithm having a negativ characteristic, (for the extraction of a root;) if the characteristic is exactly divisible, divide the decimal part and the characteristic separately:

$$\overline{12}.690\ 196\ 080\ 028,5 \div 6$$
$$\tilde{2}.115\ 032\ 680\ 004,7$$

But if the characteristic be not so divisible, add to it a negativ quantity, which will make it divisible, and prefix to the decimal part in compensation an equal quantity positiv.

	$\overline{12}.690\ 196\ 080\ 028,5 \div 5$
Add $\tilde{3}$	$\overline{15}.$
Add 3	$3.690\ 196\ 080\ 028,5$
Quotient	$\tilde{3}.738\ 039\ 216\ 005,7$

DIFFERENT BASES.

65.—Ten is the base of the logarithmic system which we have been explaining; it is the most useful of all systems, because ten is also the base of our numerical system. These are usually calld common or vulgar, or Briggsian logarithms, but decimal logarithms would seem more appropriate.

66.—Any number might form the base of a system of logarithms, but the only other in actual use is one known as the "natural" system, having for its base the number 2.718281828459+ which is the sum of the series

$$1 + \frac{1}{1} + \frac{1}{1\times 2} + \frac{1}{1\times 2\times 3} + \frac{1}{1\times 2\times 3\times 4} + \frac{1}{1\times 2\times 3\times 4\times 5} \text{ etc.}$$

This is only used in theoretical inquiries, and is seldom of utility to the accountant.

PART II.

TABLES

FOR OBTAINING

LOGARITHMS AND ANTILOGARITHMS

TO 12 PLACES OF DECIMALS

Number	A *·*	B 1.0**	C 1.000**	D 1.00000**	E 1.0^7**	F 1.0^9**	Number
01	....	434 077 479,3	4 342 923,1	43 429,4	434,3	004,3	01
02	....	867 721 531,2	8 685 802.8	86 858,9	868,6	008,7	02
03	....	1 300 933 020,4	13 028 639,0	130 288,3	1 302,9	013,0	03
04	....	1 733 712 809,0	17 371 431,8	173 717,8	1 737,2	017,4	04
05	....	2 166 061 756,5	21 714 181,2	217 147,2	2 171,5	021,7	05
06	....	2 597 980 719,9	26 056 887,2	260 576,6	2 605,8	026,1	06
07	....	3 029 470 553,6	30 399 549,8	304 006,0	3 040,1	030,4	07
08	....	3 460 532 109,5	34 742 168,9	347 435,4	3 474,4	034,7	08
09	....	3 891 166 236,9	39 084 744,6	390 864,9	3 908,7	039,1	09
10	....	4 321 373 782,6	43 427 276,9	434 294,3	4 342,9	043,4	10
11	041 392 685 158,2	4 751 155 591,0	47 769 765,7	477 723,7	4 777,2	047,8	11
12	079 181 246 047,6	5 180 512 503,8	52 112 211,2	521 153,1	5 211,5	052,1	12
13	113 943 352 306,8	5 609 445 360,3	56 454 613,2	564 582,5	5 645,8	056,5	13
14	146 128 035 678,2	6 037 954 997,3	60 796 971,8	608 011,8	6 080,1	060,8	14
15	176 091 259 055,7	6 466 042 249,2	65 139 287,0	651 441,2	6 514,4	065,1	15
16	204 119 982 655,9	6 893 707 947,9	69 481 558,7	694 870,6	6 948,7	069,5	16
17	230 448 921 378,3	7 320 952 922,7	73 823 787,1	738 300,0	7 383,0	073,8	17
18	255 272 505 103,3	7 747 778 000,7	78 165 972,0	781 729.4	7 817,3	078,2	18
19	278 753 600 952,8	8 174 184 006,4	82 508 113,5	825 158,7	8 251,6	082,5	19
20	301 029 995 664,0	8 600 171 761,9	86 850 211,6	868 588,1	8 685,9	086,9	20
21	322 219 294 733,9	9 025 742 086,9	91 192 266,3	912 017,5	9 120,2	091,2	21
22	342 422 680 822,2	9 450 895 798,7	95 534 277,6	955 446,8	9 554,5	095,5	22
23	361 727 836 017,6	9 875 633 712,2	99 876 245,5	998 876,2	9 988,8	099,9	23
24	380 211 241 711,6	10 299 956 639,8	104 218 170,0	1 042 305,5	10 423,1	104,2	24
25	397 940 008 672,0	10 723 865 391,8	108 560 051,0	1 085 734,8	10 857,4	108,6	25
26	414 973 347 970,8	11 147 360 775,8	112 901 888,7	1 129 164,2	11 291,7	112,9	26
27	431 363 764 159,0	11 570 443 597,3	117 243 682,9	1 172 593,5	11 726,0	117,3	27
28	447 158 031 342,2	11 993 114 659,3	121 585 433,8	1 216 022,8	12 160,2	121,6	28
29	462 397 997 899,0	12 415 374 762,4	125 927 141,2	1 259 452,2	12 594,5	125,9	29
30	477 121 254 719,7	12 837 224 705,2	130 268 805,2	1 302 881,5	13 028,8	130,3	30
31	491 361 693 834,3	13 258 665 283,5	134 610 425,9	1 346 310,8	13 463,1	134,6	31
32	505 149 978 319,9	13 679 697 291,2	138 952 003,1	1 389 740,1	13 897,4	139,0	32
33	518 513 939 877,9	14 100 321 519,6	143 293 536,9	1 433 169,4	14 331,7	143,3	33
34	531 478 917 042,3	14 520 538 757,9	147 635 027,3	1 476 598,7	14 766,0	147,7	34
35	544 068 044 350,3	14 940 349 792,9	151 976 474,3	1 520 028,0	15 200,3	152,0	35
36	556 302 500 767,3	15 359 755 409,2	156 317 878,0	1 563 457,3	15 634,6	156,3	36
37	568 201 724 067,0	15 778 756 389,0	160 659 238,2	1 606 886,6	16 068,9	160,7	37
38	579 783 596 616,8	16 197 353 512,4	165 000 555,0	1 650 315,9	16 503,2	165,0	38
39	591 064 607 026,5	16 615 547 557,2	169 341 828,4	1 693 745,2	16 937,5	169,4	39
40	602 059 991 328,0	17 033 339 298,8	173 683 058,5	1 737 174,5	17 371,8	173,7	40
41	612 783 856 719,7	17 450 729 510,5	178 024 245,1	1 780 603,7	17 806,1	178,1	41
42	623 249 290 397,9	17 867 718 963,5	182 365 388,3	1 824 033,0	18 240,4	182,4	42
43	633 468 455 579,6	18 284 308 426,5	186 706 488,2	1 867 462,3	18 674,7	186,7	43
44	643 452 676 486,2	18 700 498 666,2	191 047 544,7	1 910 891,5	19 109,0	191,1	44
45	653 212 513 775,3	19 116 290 447,1	195 388 557,7	1 954 320,8	19 543,3	195,4	45
46	662 757 831 681,6	19 531 684 531,3	199 729 527,4	1 997 750,0	19 977,5	199,8	46
47	672 097 857 935,7	19 946 681 678,8	204 070 453,7	2 041 179,3	20 411,8	204,1	47
48	681 241 237 375,6	20 361 282 647,7	208 411 336,6	2 084 608,5	20 846,1	208,5	48
49	690 196 080 028,5	20 775 488 193,6	212 752 176,1	2 128 037,7	21 280,4	212,8	49

Number	A * · *	B 1.0**	C 1.000**	D 1.00000**	E 1.0⁷**	F 1.0⁹**	Number
50	698 970 004 336,0	21 189 299 069,9	217 092 972,2	2 171 467,0	21 714,7	217,1	50
51	707 570 176 097,9	21 602 716 028,2	221 433 725,0	2 214 896,2	22 149,0	221,5	51
52	716 003 343 634,8	22 015 739 817,7	225 774 434,3	2 258 325,4	22 583,3	225,8	52
53	724 275 869 600,8	22 428 371 185,5	230 115 100,3	2 301 754,7	23 017,6	230,2	53
54	732 393 759 823,0	22 840 610 876,5	234 455 722,9	2 345 183,9	23 451,9	234,5	54
55	740 362 689 494,2	23 252 459 633,7	238 796 302,1	2 388 613,1	23 886,2	238,9	55
56	748 188 027 006,2	23 663 918 197,8	243 136 837,9	2 432 042,3	24 320,5	243,2	56
57	755 874 855 672,5	24 074 987 307,4	247 477 330,3	2 475 471,5	24 754,8	247,5	57
58	763 427 993 562,9	24 485 667 699,2	251 817 779,4	2 518 900,7	25 189,1	251,9	58
59	770 852 011 642,1	24 895 960 107,5	256 158 185,1	2 562 329,9	25 623,4	256,2	59
60	778 151 250 383,6	25 305 865 264,8	260 498 547,4	2 605 759,1	26 057,7	260,6	60
61	785 329 835 010,8	25 715 383 901,3	264 838 866,3	2 649 188,3	26 492,0	264,9	61
62	792 391 689 498,3	26 124 516 745,5	269 179 141,9	2 692 617,4	26 926,3	269,3	62
63	799 340 549 453,6	26 533 264 523,3	273 519 374,0	2 736 046,6	27 360,6	273,6	63
64	806 179 973 983,9	26 941 627 959,0	277 859 562,8	2 779 475,8	27 794,8	277,9	64
65	812 913 356 642,9	27 349 607 774,8	282 199 708,3	2 822 905,0	28 229,1	282,3	65
66	819 543 935 541,9	27 757 204 690,6	286 539 810,3	2 866 334,1	28 663,4	286,6	66
67	826 074 802 700,8	28 164 419 424,5	290 879 869,0	2 909 763,3	29 097,7	291,0	67
68	832 508 912 706,2	28 571 252 692,5	295 219 884,3	2 953 192,4	29 532,0	295,3	68
69	838 849 090 737,3	28 977 705 208,8	299 559 856,2	2 996 621,6	29 966,3	299,7	69
70	845 098 040 014,3	29 383 777 685,2	303 899 784,8	3 040 050,7	30 400,6	304,0	70
71	851 258 348 719,1	29 789 470 831,9	308 239 670,0	3 083 479,9	30 834,9	308,3	71
72	857 332 496 431,3	30 194 785 356,8	312 579 511,8	3 126 909,0	31 269,2	312,7	72
73	863 322 860 120,5	30 599 721 966,0	316 919 310,3	3 170 338,1	31 703,5	317,0	73
74	869 231 719 731,0	31 004 281 363,5	321 259 065,4	3 213 767,3	32 137,8	321,4	74
75	875 061 263 391,7	31 408 464 251,6	325 598 777,1	3 257 196,4	32 572,1	325,7	75
76	880 813 592 280,8	31 812 271 330,4	329 938 445,5	3 300 625,5	33 006,4	330,1	76
77	886 490 725 172,5	32 215 703 298,0	334 278 070,5	3 344 054,6	33 440,7	334,4	77
78	892 094 602 690,5	32 618 760 850,7	338 617 652,2	3 387 483,7	33 875,0	338,7	78
79	897 627 091 290,4	33 021 444 682,9	342 957 190,4	3 430 912,9	34 309,3	343,1	79
80	903 089 986 991,9	33 423 755 486,9	347 296 685,4	3 474 342,0	34 743,6	347,4	80
81	908 485 018 878,6	33 825 693 953,3	351 636 136,9	3 517 771,1	35 177,9	351,8	81
82	913 813 852 383,7	34 227 260 770,6	355 975 545,1	3 561 200,2	35 612,1	356,1	82
83	919 078 092 376,1	34 628 456 625,3	360 314 910,0	3 604 629,2	36 046,4	360,5	83
84	924 279 286 061,9	35 029 282 202,4	364 654 231,5	3 648 058,3	36 480,7	364,8	84
85	929 418 925 714,3	35 429 738 184,5	368 993 509,6	3 691 487,4	36 915,0	369,2	85
86	934 498 451 243,6	35 829 825 252,8	373 332 744,4	3 734 916,5	37 349,3	373,5	86
87	939 519 252 618,6	36 229 544 086,3	377 671 935,8	3 778 345,6	37 783,6	377,8	87
88	944 482 672 150,2	36 628 895 362,2	382 011 083,8	3 821 774,6	38 217,9	382,2	88
89	949 390 006 644,9	37 027 879 755,8	386 350 188,6	3 865 203,7	38 652,2	386,5	89
90	954 242 509 439,3	37 426 497 940,6	390 689 249,9	3 908 632,7	39 086,5	390,9	90
91	959 041 392 321,1	37 824 750 588,3	395 028 267,9	3 952 061,8	39 520,8	395,2	91
92	963 787 827 345,6	38 222 638 368,7	399 367 242,6	3 995 490,9	39 955,1	399,6	92
93	968 482 948 553,9	38 620 161 949,7	403 706 173,9	4 038 919,9	40 389,4	403,9	93
94	973 127 853 599,7	39 017 321 997,4	408 045 061,8	4 082 348,9	40 823,7	408,2	94
95	977 723 605 288,8	39 414 119 176,1	412 383 906,5	4 125 778,0	41 258,0	412,6	95
96	982 271 233 039,6	39 810 554 148,4	416 722 707,7	4 169 207,0	41 692,3	416,9	96
97	986 771 734 266,2	40 206 627 574,7	421 061 465,6	4 212 636,0	42 126,6	421,3	97
98	991 226 075 692,5	40 602 340 114,1	425 400 180,2	4 256 065,1	42 560,9	425,6	98
99	995 635 194 597,5	40 997 692 423,5	429 738 851,4	4 299 494,1	42 995,2	430,0	99

1 + i	Logarithm	1 + i	Logarithm
1.00125	000 542 529 092 294	1.01375	005 930 867 219 212
1.0015	000 650 953 629 595	1.014	006 037 954 997 317
1.00175	000 759 351 104 737	1.01425	006 145 016 376 364
1.002	000 867 721 531 227	1.0145	006 252 051 369 365
1.00225	000 976 064 922 559	1.01475	006 359 059 989 323
1.0025	001 084 381 292 220	1.015	006 466 042 249 232
1.00275	001 192 670 653 684	1.01525	006 572 998 162 075
1.003	001 300 933 020 418	1.0155	006 679 927 740 826
1.00325	001 409 168 405 876	1.01575	006 786 830 998 449
1.0035	001 517 376 823 504	1.016	006 893 707 947 900
1.00375	001 625 558 286 737	1.01625	007 000 558 602 125
1.004	001 733 712 809 001	1.0165	007 107 382 974 057
1.00425	001 841 840 403 709	1.01675	007 214 181 076 625
1.0045	001 949 941 084 268	1.017	007 320 952 922 745
1.00475	002 058 014 864 072	1.01725	007 427 698 525 323
1.005	002 166 061 756 508	1.0175	007 534 417 897 258
1.00525	002 274 081 774 949	1.01775	007 641 111 051 437
1.0055	002 382 074 932 761	1.018	007 747 778 000 740
1.00575	002 490 041 243 299	1.01825	007 854 418 758 035
1.006	002 597 980 719 909	1.0185	007 961 033 336 183
1.00625	002 705 893 375 925	1.01875	008 067 621 748 033
1.0065	002 813 779 224 673	1.019	008 174 184 006 426
1.00675	002 921 638 279 469	1.01925	008 280 720 124 194
1.007	003 029 470 553 618	1.0195	008 387 230 114 159
1.00725	003 137 276 060 415	1.01975	008 493 713 989 132
1.0075	003 245 054 813 147	1.02	008 600 171 761 918
1.00775	003 352 806 825 089	1.02025	008 706 603 445 309
1.008	003 460 532 109 506	1.0205	008 813 009 052 089
1.00825	003 568 230 679 656	1.02075	008 919 388 595 035
1.0085	003 675 902 548 784	1.021	009 025 742 086 910
1.00875	003 783 547 730 127	1.02125	009 132 069 540 472
1.009	003 891 166 236 911	1.0215	009 238 370 968 466
1.00925	003 998 758 082 352	1.02175	009 344 646 383 631
1.0095	004 106 323 279 658	1.022	009 450 895 798 694
1.00975	004 213 861 842 026	1.02225	009 557 119 226 374
1.01	004 321 373 782 643	1.0225	009 663 316 679 379
1.01025	004 428 859 114 686	1.02275	009 769 488 170 411
1.0105	004 536 317 851 323	1.023	009 875 633 712 160
1.01075	004 643 750 005 712	1.02325	009 981 753 317 307
1.011	004 751 155 591 001	1.0235	010 087 846 998 524
1.01125	004 858 534 620 329	1.02375	010 193 914 768 475
1.0115	004 965 887 106 823	1.024	010 299 956 639 812
1.01175	005 073 213 063 604	1.02425	010 405 972 625 180
1.012	005 180 512 503 780	1.0245	010 511 962 737 214
1.01225	005 287 785 440 451	1.02475	010 617 926 988 539
1.0125	005 395 031 886 706	1.025	010 723 865 391 773
1.01275	005 502 251 855 626	1.02525	010 829 777 959 522
1.013	005 609 445 360 280	1.0255	010 935 664 704 385
1.01325	005 716 612 413 731	1.02575	011 041 525 638 950
1.0135	005 823 753 029 028	1.026	011 147 360 775 797

TABLE OF INTEREST RATIOS—Continued

$1 + i$	Logarithm
1.02625	011 253 170 127 497
1.0265	011 358 953 706 611
1.02675	011 464 711 525 690
1.027	011 570 443 597 278
1.02725	011 676 149 933 909
1.0275	011 781 830 548 107
1.02775	011 887 485 452 387
1.028	011 993 114 659 257
1.02825	012 098 718 181 213
1.0285	012 204 296 030 743
1.02875	012 309 848 220 326
1.029	012 415 374 762 433
1.02925	012 520 875 669 524
1.0295	012 626 350 954 050
1.02975	012 731 800 628 455
1.03	012 837 224 705 172
1.0305	013 047 996 115 232
1.031	013 258 665 283 517
1.0315	013 469 232 309 170
1.032	013 679 697 291 193
1.0325	013 890 060 328 439
1.033	014 100 321 519 621
1.0335	014 310 480 963 307
1.034	014 520 538 757 924
1.0345	014 730 495 001 753
1.035	014 940 349 792 937
1.0355	015 150 103 229 471
1.036	015 359 755 409 214
1.0375	015 988 105 384 130
1.038	016 197 353 512 439
1.039	016 615 547 557 177
1.04	017 033 339 298 780
1.041	017 450 729 510 536
1.0425	018 076 063 645 795
1.043	018 284 308 426 531
1.044	018 700 498 666 243
1.045	019 116 290 447 073
1.046	019 531 684 531 255
1.0475	020 154 031 638 333
1.048	020 361 282 647 708
1.049	020 775 488 193 558
1.05	021 189 299 069 938
1.055	023 252 459 633 711
1.06	025 305 865 264 770
1.065	027 349 607 774 757
1.07	029 383 777 685 210
1.075	031 408 464 251 624
1.08	033 423 755 486 950
1.09	037 426 497 940 624
1.10	041 392 685 158 225

TABLE OF SUB-RECIPROCALS (Art. 51)

Initial Figures	Sub-reciprocal
10	90
11	83
12	76
13	71
14	66
15	62
16	58
17	55
18	52
19	50
20	47
21	45
22	43
23	41
24	40
25	38
26	37
27	35
28	34
29	33
30	32
31	31
32	30
33	29
34	28
35–36	27
37	26
38–39	25
40	24
41–42	23
43–44	22
45–46	21
47–49	20
50–51	19
52–54	18
55–57	17
58–61	16
62–65	15
66–70	14
71–75	13
76–82	12
83–89	11
90	1
..	..
..	..
..	..
..	..
..	..
..	..
..	..

1	2	3		4	5	6		7	8	9
001	002	003		004	005	006		007	008	009
002	004	006		008	010	012		014	016	018
003	006	009		012	015	018		021	024	027
004	008	012		016	020	024		028	032	036
005	010	015		020	025	030		035	040	045
006	012	018		024	030	036		042	048	054
007	014	021		028	035	042		049	056	063
008	016	024		032	040	048		056	064	072
009	018	027		036	045	054		063	072	081
010	020	030		040	050	060		070	080	090
011	022	033		044	055	066		077	088	099
012	024	036		048	060	072		084	096	108
013	026	039		052	065	078		091	104	117
014	028	042		056	070	084		098	112	126
015	030	045		060	075	090		105	120	135
016	032	048		064	080	096		112	128	144
017	034	051		068	085	102		119	136	153
018	036	054		072	090	108		126	144	162
019	038	057		076	095	114		133	152	171
020	040	060		080	100	120		140	160	180
021	042	063		084	105	126		147	168	189
022	044	066		088	110	132		154	176	198
023	046	069		092	115	138		161	184	207
024	048	072		096	120	144		168	192	216
025	050	075		100	125	150		175	200	225
026	052	078		104	130	156		182	208	234
027	054	081		108	135	162		189	216	243
028	056	084		112	140	168		196	224	252
029	058	087		116	145	174		203	232	261
030	060	090		120	150	180		210	240	270
031	062	093		124	155	186		217	248	279
032	064	096		128	160	192		224	256	288
033	066	099		132	165	198		231	264	297
034	068	102		136	170	204		238	272	306
035	070	105		140	175	210		245	280	315
036	072	108		144	180	216		252	288	324
037	074	111		148	185	222		259	296	333
038	076	114		152	190	228		266	304	342
039	078	117		156	195	234		273	312	351
040	080	120		160	200	240		280	320	360
041	082	123		164	205	246		287	328	369
042	084	126		168	210	252		294	336	378
043	086	129		172	215	258		301	344	387
044	088	132		176	220	264		308	352	396
045	090	135		180	225	270		315	360	405
046	092	138		184	230	276		322	368	414
047	094	141		188	235	282		329	376	423
048	096	144		192	240	288		336	384	432
049	098	147		196	245	294		343	392	441

1	2	3		4	5	6		7	8	9
050	100	150		200	250	300		350	400	450
051	102	153		204	255	306		357	408	459
052	104	156		208	260	312		364	416	468
053	106	159		212	265	318		371	424	477
054	108	162		216	270	324		378	432	486
055	110	165		220	275	330		385	440	495
056	112	168		224	280	336		392	448	504
057	114	171		228	285	342		399	456	513
058	116	174		232	290	348		406	464	522
059	118	177		236	295	354		413	472	531
060	120	180		240	300	360		420	480	540
061	122	183		244	305	366		427	488	549
062	124	186		248	310	372		434	496	558
063	126	189		252	315	378		441	504	567
064	128	192		256	320	384		448	512	576
065	130	195		260	325	390		455	520	585
066	132	198		264	330	396		462	528	594
067	134	201		268	335	402		469	536	603
068	136	204		272	340	408		476	544	612
069	138	207		276	345	414		483	552	621
070	140	210		280	350	420		490	560	630
071	142	213		284	355	426		497	568	639
072	144	216		288	360	432		504	576	648
073	146	219		292	365	438		511	584	657
074	148	222		296	370	444		518	592	666
075	150	225		300	375	450		525	600	675
076	152	228		304	380	456		532	608	684
077	154	231		308	385	462		539	616	693
078	156	234		312	390	468		546	624	702
079	158	237		316	395	474		553	632	711
080	160	240		320	400	480		560	640	720
081	162	243		324	405	486		567	648	729
082	164	246		328	410	492		574	656	738
083	166	249		332	415	498		581	664	747
084	168	252		336	420	504		588	672	756
085	170	255		340	425	510		595	680	765
086	172	258		344	430	516		602	688	774
087	174	261		348	435	522		609	696	783
088	176	264		352	440	528		616	704	792
089	178	267		356	445	534		623	712	801
090	180	270		360	450	540		630	720	810
091	182	273		364	455	546		637	728	819
092	184	276		368	460	552		644	736	828
093	186	279		372	465	558		651	744	837
094	188	282		376	470	564		658	752	846
095	190	285		380	475	570		665	760	855
096	192	288		384	480	576		672	768	864
097	194	291		388	485	582		679	776	873
098	196	294		392	490	588		686	784	882
099	198	297		396	495	594		693	792	891

SPECIMEN OF RULED PAPER

RECOMMENDED FOR USE WITH THE FOREGOING TABLES.

PART III.

The Doctrine of Interest

PART III.

THE DOCTRIN OF INTEREST.

INTEREST.

67.—**Interest**, mathematically considerd, is the increase of an indettedness by lapse of time. The rate of such increase varies with circumstances, * and is subject to bargaining ; the resulting contract, exprest or implied, must embody the following terms:

Principal. The number of units of value (dollars, pounds, francs, marks, etc.,) originally loand or invested.

Interest Rate. The fraction which is added to each unit by the lapse of one unit of time ; usually a small decimal.

Frequency. The length of the unit of time, measured in years, months or days.

Time. The number of units of time during which the indettedness is to continue.

68.—As each dollar increases just as much as every other dollar, it is best at first to consider the principal as *one dollar* and when the proper function thereof has been calculated, to multiply it by the *number of* dollars.

69.—The **interest rate** is usually spoken of as so much per cent per period or term. "6% per annum" means an increase of .06 for each term of a year. We will designate the interest rate by the letter i; as, $i = .06$. At the end of one term the increast indettedness is $1 + i$, (1.06), a very important quantity in computation.

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* For discussion of the causes for higher or lower interest rates, see *The Rate of Interest*, by Prof. Irving Fisher.

70.—**Punctual Interest.** The usual contract is that the increase shall be paid off in cash at the end of each period, restoring the principal to its original quantity. Let c denote the cash payment; then $1 + i - c = 1$; and the second term would repeat the same process. The payment of cash *for* interest must not be regarded as the interest: it is a cancellation of part of the increast principal. Many persons, and even courts, have been misled by the old definition of interest, "money paid for the use of money," into treating uncollected or unmatured interest as a nullity, tho secured precisely in the same way as the principal.

71.—But the interest money may not be paid exactly at the end of each term, either in violation of the contract or by a special clause permitting it to run on, or by the det being assigned to a third party at a price which modifies the true interest rate. In this case the question arises: how shall the interest be computed for the following periods? This gives rise to a distinction between *simple* and *compound* interest.

72.—**Simple Interest.** During the second period, altho the borrower has in his hands an increast principal, $1 + i$, he is at simple interest only charged with interest on 1, and has the free use of i, which tho small has an earning power proportionate to that of 1. His indettedness at the end of the second term is $1 + 2i$, and thereafter $1 + 3i$, $1 + 4i$, etc. After the first period he is *not* charged with the agreed percentage of the sum actually employed by him, and this to the detriment of the creditor. For any scientific calculation, simple interest is impossible of application.

73.—**Compound Interest.** The indettedness at the end of the first period is $1 + i$, and up to this point *punctual*, *simple* and *compound* interest coincide. But in compound interest the fact is recognized that the increast principal, $1 + i$, is *all* subject to interest during the next period, and that the det increases by *geometrical* progression, not arithmetical. The increase from 1 to $1 + i$ is regarded, not as an addition of i to 1, but as a multiplication of 1 by the ratio of increase $(1 + i)$. We shall designate the ratio of increase by r when convenient, altho this is merely an abbreviation of $1 + i$, and the two expressions are at all times interchangeable.

74.—For the second period, $1 + i$ is the actual and equitable principal, and it should be again increast in the ratio $1 + i$. The total indettedness at the end of the second period is therefore $1 \times (1 + i) \times (1 + i) = (1 + i)^2 = r^2$. At the end of the third period it will have become r^3, and at the end of term No. t, r^t.

THE AMOUNT.

75.—The sum to which $1 will have increast at compound interest at i (or $100i$ per cent.) in t periods, is called the Amount, and will be designated as s. We then have the following equation:

$$s = r^t = (1 + i)^t$$

76.—To find the amount of one dollar, raise the ratio to a power whose exponent is the number of periods.

77.—The logarithm of the ratio of increase is the most important logarithm for interest calculations. If the interest rate does not exceed two figures, the logarithm will be found in full in col. B, TF. For convenience we will designate it by a capital letter L. Thus, if $i = .065$, L will be found opposit 65 in B. If $i = .065$; log. $r = L = .027\ 349\ 607\ 774{,}8$.

78.—As powers are found by multiplying the logarithm, L must be multiplied by t.

$$r^t \qquad nl \qquad tL$$

79.—To find the amount, multiply the logarithm of the ratio by the number of periods, and the corresponding number will be the amount of $1.

80.—Let the interest rate be 3.5% per annum, payable annually, what will be the amount of $1 at the end of 100 years? Turning to col. B, TF, we find opposit B 35, (or 1.035) the logarithm .014 940 349 792,9.

$$L = \ .014\ 940\ 349\ 792{,}9$$

$$t = 100 \qquad\qquad tL = 1.494\ 034\ 979\ 29$$

From the characteristic 1, it appears that the amount will be in the tens of dollars ; and as the decimal part of the logarithm is a little more than that which is opposit 31 we know that the amount is $31 and some cents. Thus a rough idea of the amount may be gaind almost instantly.

81.—To obtain a more accurate value and one which will be sufficiently near for a large principal, we proceed as follows:

82.—In the first place we can only obtain ten correct figures from 100L. The final figure 9 is never perfect; it may be 8.51 or 9.49 or anywhere between. We must, therefore, use only eleven in the logarithm and finally get ten in the number.

(?)	*nl*	494 034 979 29
A 31		491 361 693 83
		2 673 285 46
B 06		2 597 980 72
		75 304 74
C 17		73 823 79
		1 480 95
D 34		1 476 60
		4 35
E 10		4 34
F 2		1

	3 1
B 06	1 8 6
	3 1 1 8 6 . . . .
C 1	3 1 1 8 6
7	2 1 8 3 0 2
	3 1 1 9 1 3 0 1 6 2
D 3	9 3 5 7
4	1 2 4 8
	3 1 1 9 1 4 0 7 6 7
E 10	3 1
	3 1 1 9 1 4 0 7 9 8 $= s\ (t\ 100,\ i\ .03)$

83.—In order to give accurate results up to twelve figures for one hundred interest terms, we have provided on page 32 a special table of the logarithms of the 150 interest ratios $(1 + i)$ which most frequently occur, calculated to 15 places, which allows two places for loss in multiplication.

THE PRESENT WORTH.

84.—The sum which if now invested at i will in t periods amount to \$1 is evidently less than \$1. It is in the same proportion to 1 as 1 is to s. Designating the present worth by p, we have

$$p : 1 :: 1 : s$$

$$\text{or } p = \frac{1}{s} = s^{-1}$$

or the amount and the present worth are reciprocals of each other.

A series of amounts reads

$$1, r^1, r^2, r^3, r^4, r^5, \text{etc.}$$

A series of present worths reads

$$1, r^{-1}, r^{-2}, r^{-3}, r^{-4}, r^{-5}, \text{etc.}$$

Reversing the latter series and connecting it with the former we have a continuous series in geometrical progression:

$$r^{-5}, r^{-4}, r^{-3}, r^{-2}, r^{-1}, 1, r^1, r^2, r^3, r^4, r^5.$$

Using 1.03 as the ratio, the series becomes

r^{-5}	.86260878
r^{-4}	.88848705
r^{-3}	.91514166
r^{-2}	.94259591
r^{-1}	.97087379
r^0	1.
r^1	1.03
r^2	1.0609
r^3	1.092727
r^4	1.12550881
r^5	1.15927407

In this series, which might be extended indefinitly upward and downward, every term is a present worth of any which follows it and an amount of each which precedes it. .86260878 is the present worth at 10 interest periods of 1.15927407; 1.12550881 is the amount at eight periods of .88848705.

85.—If any term be multiplied by 1.03, the product will be the next following term; if it be divided by 1.03 or (which is the same thing) be multiplied by .97087379, the product will be the next preceding term.

86.—To find the logarithm of the present worth, subtract the logarithm of the amount (for the same time) from zero.

In the preceding example, but using L from the 15 place table

s	nl	$tL = 1.494\ 034\ 979\ 293{,}7$
$p = 1/s$	nl	$-tL = \bar{2}.505\ 965\ 020\ 706{,}3$
	A 32	505 149 978 319,9
		815 042 386,4
	B 01	434 077 479,3
		380,964 907,1
	C 87	377 671 935,8
		3 292 971,3
	D 75	3 257 196,4
		35 774,9
	E 82	35 612,1
		162,8
	F 37	160,7
	G 49	2,1

	3 2
B 01	0 3 2
	3 2 0 3 2 . . . .
C 8	2 5 6 2 5 6
7	2 2 4 2 2 4
	3 2 0·5 9 8 6 7 8 4 . . .
D 7	2 2 4 4 1 9 0 7
5	1 6 0 2 9 9 3
	3 2 0 6 0 1 0 8 2 8 9 0 0
E 8	2 5 6 4 8 1
2	6 4 1 2
F 3	9 6 2
7	2 2
G 4	2
9	
	3 2 0 6 0 1 1 0 9 2 7 7 9

$.0\,3\,2\,0\,6\,0\,1\,1\,0\,9\,2\,8 = p\ (1.035)^{100}$ to 12 places.

87.—That the amount and the present worth are correct reciprocals may be tested by multiplying them together. Taking a few figures of each we have

$$
\begin{array}{r|r}
 & 31.1914 \\
\hline
.03 & 935742 \\
2 & 62383 \\
.06 & 1871 \\
.01 & 1 \\
\hline
\multicolumn{2}{c}{1.00000}
\end{array}
$$

Every pair of reciprocals gives a product of 1.

THE COMPOUND INTEREST AND DISCOUNT.

88.—We have hitherto used the word "interest" abstractly as denoting that force or principle which effects the increase of the amount of an indettedness as time goes on. The interest-increment which is thus added is also frequently called "*the* Interest," which may be written with a capital letter.

89.—If we take the original principal away from the amount, we evidently have the Interest. For a single period

$$i = 1 + i - 1 = r - 1.$$

When there are more than one period it is the compound Interest, obtaind in the same way and represented by a capital $\mathrm{I} = (1 + i)^t - 1 = r^t - 1 = S - 1$.

Thus the compound Interest of \$1 at 3% per period for 100 periods is \$31.19 — 1.00 = \$30.19. For two periods it is 1.0609 — 1 = 0.0609.

90.—In the opposit case of a present worth there is a diminution of the principal. The present worth of \$1 at 3 per cent, one period, is .97087379; the Discount is not .03, but .02912621, the true principal being not \$1, but .97087379, which × .03 = .02912621. Representing the simple Discount by d, we have $d = 1 - p = i \times p = i/s$.

91.—If there be more than one term involvd, it is compound Discount, which will be represented by D. Thus, at 3 per cent for 5 periods D = 1 — .86261 = .13739. D is also the present worth of the compound Interest for the same time. .15927 × .86261 = .13739.

In general $D = 1 - p = Ip = I/s$.

92.—Thus we see that the variance from par ($1) is called compound Interest or compound Discount, according as regarded from the past or the future point of view and that their properties are as follows:

$$I = r^t - 1$$
$$D = 1 - r^{-t}$$

and their relation is $D = pI$; or $I = sD$.

FINDING TIME OR RATE.

93.—By time we mean the number of periods, terms or intervals, and by this number the logarithm of the interest-ratio is multiplied to produce the logarithm of the amount.

$$(t \times L) \; ln \; s$$
$$t \times log \, (1 + i) = log \, s$$

94.—If the amount is known and the rate, but the number of periods unknown, we can transform the above equation into this:

$$t = \frac{log \; s}{L}$$

95.—At .03 interest, in how many periods will $1 amount to $2, or how long will it take a sum to double itself?

$$log \; s = log \; 2 = .3010299956640$$
$$L = log \; 1.03 = .0128372247052$$

Using only seven places

```
.0128372).3010300(23.47
          256744
           442860
           385116
            57744
            49349
             8395
```

The money will double in 24 periods, as it is not quite doubled at 23.

96.—How many periods must a det of $1 be deferd to be worth now 30 cents, at 3½%?

$$\begin{aligned} Log\,1.035 &= .01494035 \\ Log\,1.035^{-1} &= \bar{1}.98505965 \\ Log\,.30 &= \bar{1}.47712125 \end{aligned}$$

$$\begin{array}{l} \bar{1}.98505965)\ \bar{1}.47712125(\\ -.01494035)-.52287875(34.9997 \\ \qquad\qquad\qquad 4482105 \\ \qquad\qquad\qquad \overline{\ \ 7466825} \\ \qquad\qquad\qquad\ \ 5976140 \\ \qquad\qquad\qquad \overline{\ \ 1490685} \\ \qquad\qquad\qquad\ \ 1344631 \\ \qquad\qquad\qquad \overline{\ \ \ \ 146054} \\ \qquad\qquad\qquad\ \ \ \ 134463 \\ \qquad\qquad\qquad \overline{\ \ \ \ \ \ 11591} \end{array}$$

Practically 35 periods.

For convenience in division, the minus sign is made to extend over the entire logarithms. Then, as both divisor and dividend are of the same sign, the quotient is positiv.

97.—If the rate be unknown, the equation $t \times \log.\ (1 + i) = \log.\ s$ may again be transformed to

$$\log.\ (1 + i) = \frac{\log.\ s}{t}.$$

98.—20 periods having elapst and the amount of $1 being now $3.20713547, what is the rate?

$$\begin{aligned} \log.\ 3.20713547 &= .506117303 \\ .506117303/20 &= .025305865\ ln\ 1.06 \\ 1 + i = 1.06\ &\therefore\ i = .06 \end{aligned}$$

THE ANNUITY.

99.—We have now investigated the two fundamental problems in compound interest: viz., to find the amount of a present worth, and to find the present worth of an amount. The next question is a more complex one: to find the amount and the present worth of a *series* of payments. If these payments are irregular as to time, amount and rate of interest, the only way is to make as many separate computations as there are sums and then add them together. But if the sums, times and rate are uniform, we can devise a method for finding the amount or present worth at one operation.

100.—**Annuity.** A series of payments of like amount, made at regular periods, is called an annuity, even though the period be not annual, but a half year, a quarter or any other length of time. Thus, if an agreement is made for the following payments:

On Sept. 9 1904	$100.
On March 9 1905	100.
On Sept. 9 1905	100.
and on March 9 1906	100.

this would be an annuity of $100 per period, terminating after 4 periods. It is required to find on March 9, 1904, assuming the rate of interest as 3% per period: First, what will be the total amount to which the annuity will have accumulated on March 9, 1906; second, what is now, on March 9, 1904, the present worth of this series of future sums? It is evident that the answer to the first question will be greater than $400, and that the answer to the second question will be less than $400.

AMOUNT OF AN ANNUITY.

101.—It is easy, in this case, to find the separate amounts of the payments, for the number of terms is very small, and we have already computed the corresponding values of $1.00.

The last $100 will have no accumulation, and will be merely..................................	$100.
The third $100 will have earned in one period $3.00, and will amount to..........................	103.
The second $100 will amount to.................	106.09
The first $100 (rounded off at cents) will amount to	109.27
and the total amount will be.................	$418.36

102.—If, however, there were 50 terms instead of 4, the work of computing these 50 separate amounts, even by the use of logarithms, would be very tedious.

103.—Let us write down the successiv amounts of $1.00 under one another:

a
Amounts of $1.
1.00
1.03
1.0609
1.092727

104.—Now, as we have the right to take any principal we choose and multiply it by the number indicating the value of \$1.00, let us assume one dollar and three cents, and multiply each of the above figures by 1.03, setting the products in a second colum:

a. Amounts of \$1.00	*b.* Amounts of \$1.03	*c.* Amounts of \$0.03
1.00	1.03	
1.03	1.0609	
1.0609	1.092727	
1.092727	1.12550881	

105.—Our object in doing this was by subtracting colum *a* from *b* to find the amount of an annuity of three cents. Before subtracting, we have the right to throw out any numbers which are identical in the two colums. Expunging these like quantities, we have left only the following:

a. Annuity of \$1.00	*b.* Annuity of \$1.03		*c.* Annuity of \$0.03
1.00	....		1.12550881
....	....	less	1.00
....	....		
....	1.12550881	Amount	0.12550881

That is, an annuity of *three cents* will amount, under the conditions assumed, to twelve cents and the decimal 550881. Therefore, an annuity of *one cent* will amount to one-third of .12550881 or .04183627. An annuity of \$1.00 will amount to 100 times as much, or \$4.183627, which agrees exactly with the result obtained by addition, in Article 45.

106.—The number .12550881 (obtaind by subtracting 1.00 from 1.12550881) is actually the *compound Interest* for the given rate and time, and the number .03 is the *single Interest*; the amount of the annuity of \$1.00 is .12550881 ÷ .03 = 4.183627. This suggests another way of looking at it. The compound Interest up to any time is really the *amount* of a smaller annuity, one of three cents instead of a dollar, constructed on exactly the same plan, and used as a model.

107.—Rule. To find the *amount* of an annuity of \$1.00 for a given time and rate, divide the compound Interest by a single Interest, both exprest decimally.

108.—Let S and P represent the amount and the present worth, not of a single \$1.00, but of an annuity of \$1, then $S = I \div i$.

Exprest in symbols the reasoning would be this:

Amount of annuity of $1 = r^{t-1} + \ldots\ldots r^3 + r^2 + r + 1$ (a)

Multiplying by r.

Amount of annuity of $1 + i = r^t + r^{t-1} \ldots r^4 + r^3 + r^2 + r$ (b)

Subtracting (a) $\qquad r^{t-1} \ldots\ldots r^3 + r^2 + r + 1$

Amount of annuity of $i = r^t \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 - 1$ (c)

$= r^t - 1 = I$

Amount of annuity of $1 = S \qquad = I/i$

109.—If the number of periods were 50, insted of 4, the advantage of this process, with the use of logarithms, will be very evident.

The rate being .03, the logarithm of the ratio, or

$$L = .012\ 837\ 224\ 705\ 172$$

$$50\ L = .641\ 861\ 235\ 258,6$$

Factors, A 43 B 19 C 50 D 34 E 60 F 10 G 14

$$s = 4\ .\ 3\ 8\ 3\ 9\ 0\ 6\ 0\ 1\ 8\ 7\ 6$$

$$-1$$

$$I = 3\ .\ 3\ 8\ 3\ 9\ 0\ 6\ 0\ 1\ 8\ 7\ 6$$

$$I \div .03 = 1\ 1\ 2\ .\ 7\ 9\ 6\ 8\ 6\ 7\ 2\ 9\ 2 = S$$

Compare this with the difficulty of finding the result by arithmetic for even ten periods.

PRESENT WORTH OF AN ANNUITY.

110.—To find the present worth of an annuity, we can, of course, find the present worth of each payment and add them together; but it will evidently save a great deal of labor if we can derive the present worth immediately, as we have learnd to do with the amount.

111.—The like course of reasoning will give us the result. Take the four numbers representing the present worths of \$1.00 at 4, 3, 2 and 1 periods respectivly, and multiply each by 1.03.

a. Present Worth of Annuity of \$1.00	*b.* Present Worth of Annuity of \$1.03
.888487	.915142
.915142	.942596
.942596	.970874
.970874	1.000000

Canceling all equivalents, we have

			c. Present Worth of Annuity of .03
.888487	....		
....	....		
....	....		1.000000
....	1.000000	less	.888487
			.111513

Annuity of \$1.00 = .111513 ÷ .03 = 3.71710

This is the same result (rounded up) as that obtaind by adding column *a*.

112.—But .111513 is the compound discount of \$1.00 for four periods, and we therefore construct this rule:

113.—Rule. To find the present worth of an annuity of \$1.00 for a given time and rate, divide the compound Discount for that time and rate by a single interest. Symbolically $P = D \div i$. We might give this the form $P = S \div s$, being the present worth of the *amount* of the annuity.

114.—It may assist in acquiring a clear idea of the working of an annuity, if we analyse a series of annuity payments from the point of view of the purchaser.

115.—He who invests \$3.7171 at 3%, in an annuity of 4 periods, expects to receive at each payment, besides 3% on his principal to date, a portion of that principal, and thus to have his entire principal gradually repaid.

		Principal.
His original principal is		3.7171
At the end of the first period he receives 1.00, consisting of 3% on 3.7171	.1115	
and payment on principal	.8885	.8885
leaving new principal		2.8286
(or present worth at 3 periods).		
In the next instalment	1.00	
there is interest on 2.8286	.0849	
and payment on principal	.9151	.9151
leaving new principal		1.9135
Third instalment	1.00	
Interest	.0574	
on principal	.9426	.9426
		.9709
Last instalment	1.00	
Interest	.0291	
Principal in full	.9709	.9709

Thus the annuitant has received interest in full on the principal outstanding, and has also received the entire original principal. The correctness of the basis on which we have been working is corroborated.

116.—It is usual to form a schedule showing the components of each instalment in tabular form.

Date	Total Instalment	Interest Payments	Payments on Principal	Principal Outstanding
1904 Mar. 9				3.7171
1904 Sept. 9	1.00	.1115	.8885	2.8286
1905 Mar. 9	1.00	.0849	.9151	1.9135
1905 Sept. 9	1.00	.0574	.9426	0.9709
1906 Mar. 1	1.00	.0291	.9709	0.0000
	4.00	.2829	3.7171	

117.—The payments on principal are known as amortization, which may be defined as the gradual repayment of a principal sum thru the operation of compound interest. It differs from the ordinary compound interest in this, that the new principal for each period is less than the previous one.

118.—As an example of logarithmic evaluation of an annuity, take an annuity of $1, as before, for 50 periods at the rate of .03 per period. At the beginning of the first period, what is its present worth, or what should be paid in one sum for such annuity?

$i = .03$ $r = 1.03$ nl .012 837 224 705 172 (to 15 places)

50 L = .641 861 235 258,6

As we are discounting, not accumulating, we must take the cologarithm — 50 L $\bar{1}$.358 138 764 741,4

and find the number. Factors A22 B36 C82 D08 E12 F73 G23

$p = 1.03^{-50} =$ .228 107 079 790

$D = 1 - p =$ .771 892 920 210

$D \div .03 =$ 25.729 764 007 0 $= P$

This may be proved down to maturity by amortization, the schedule beginning thus:

No.	Instalment	Payments of Interest at 3%	Payments on Principal	Principal Outstanding
				25.729 764
1	1.00	.771 893	.228 107	25.501 657
2	1.00	.765 050	.234 950	25.266 707
3	1.00	.757 901	.242 099	25.024 608
	etc.	etc.	etc.	etc.
49	1.00	.057 404	.942 596	.970 874
50	1.00	.029 126	.970 874	.000 000

119.—It may be notist that each payment on principal, or amortization for one period, is the present worth of the instalment at the *beginning* of its period. From this the instalment of amortization may be calculated at any point independently of any other figures. Thus the payment on principal in the 21st instalment of $1 is the present worth of $1.00 in 30 periods, or .411987; because at the beginning of the 21st period there were 30 instalments yet to come.

120.—It will also be notist that each amortization multiplied by 1.03 becomes the next following, these being a series of present worths; and that thus they may be derived from one another, upwards or downwards.

SPECIAL FORMS OF ANNUITY.

121.—The annuities heretofore spoken of are payable at the end of each period, and are the kind most frequently occurring. To distinguish them from other varieties they are spoken of as ordinary or immediate annuities.

122.—When the instalment (or rent) of the annuity is payable at the beginning of the period, it is called an annuity due, altho "prepaid" would seem more natural. It is evident that this is merely a question of dating. The instalments compared with those in Art. 103 are as follows:

	Immediate Annuity 4 Periods	Annuity Due 4 Periods	Immediate Annuity 5 Periods
Amounts of	1.00	1.03	1.00
	1.03	1.0609	1.03
	1.0609	1.0927	1.0609
	1.0927	1.1255	1.0927
			1.1255
			5.3091
			—1
	4.1836	4.3091	4.3091

To find the amount of an annuity *due*, for t periods, find the amount of an *immediate* annuity for $t + 1$ periods and subtract $1.

123.—In finding the present worth:

Immediate Annuity 4 Periods	Annuity Due 4 Periods	Immediate Annuity 3 Periods
.888487	.915142	.915142
.915142	.942596	.942596
.942596	.970874	.970874
.970874	1.00	2.828612
		+1.
	3.828612	3.828612

To find the present worth of an annuity *due* for t periods find the present worth of an *immediate* annuity for $t - 1$ periods and add $1.

124.—A deferd annuity is one which does not commence to run immediately, but after a certain number of periods, as an annuity of 5 terms, 4 terms deferd, which would begin at the fourth period from now and continue to the ninth inclusiv.

Its present worth is $r^4 + r^5 + r^6 + r^7 + r^8$

An annuity of the entire nine terms would be worth now

$$1 + r^1 + r^2 + r^3 + r^4 + r^5 + r^6 + r^7 + r^8$$

If from this the value of the four deferd terms be subtracted it will leave the value of the deferd annuity.

125.—To find the present worth of an annuity for m terms, deferd n terms, subtract from the value of $m + n$ terms that for n.

126.—A perpetual annuity, or a perpetuity, is one which never terminates. Its amount is infinity, but its present worth can be calculated at any certain rate of interest. If the rent of the annuity is $1 and the rate is .05, the value of the annuity is such a sum as will produce $1 at that rate or $200, being $1 / .05. The compound discount is the entire $1, being for an infinit number of terms; therefore the rule still holds: divide the compound discount by the rate of interest.

127.—Annuities at two successiv rates may occur; say 5 per cent. for 10 years and then 4 per cent. for 10 more. The second part is evidently a deferd annuity, and therefore its present worth is the same as

20 years	at	4%
less 10 years	at	4%
+ 10 years	at	5%

128.—In all these examples of annuities it has been assumed that the term or interval between payments is the same length of time as the interest-period. For example, the rate of interest may be so much per year, while the payments are half-yearly or quarterly; or there may be yearly payments while the desired interest-rate is to be on a half-yearly basis. We shall defer the treatment of these cases until the subject of nominal and effectiv rates has been discust.

129.—There may also be varying annuities, where the instalment changes by some uniform law. These seldom occur in practice. Where the change is simple, as in arithmetical progression, the annuity may be regarded as the sum of several annuities, otherwise the values must be separately calculated for each term. An annuity running for 5 terms, as follows: 13, 18, 23, 28, 33, may be regarded as (1) an annuity of 13 of 5 terms; (2) an annuity of 5, 4 terms; (3) an annuity of 5, 3 terms; (4) an annuity of 5, 2 terms; (5) a single amount of 5.

THE UNIT OF TIME.

130.—It makes no difference in the result whether each term is a year, or a month, or a day, so long as the *number* of terms (t) and the rate per term (i) are ascertaind. But unfortunately the habit has been fixt in common speech of stating the rate, not at so much per term, but so much *per annum*, even when the interest is payable or chargeable semi-annually (which is the prevalent custom), or quarterly, or monthly.

131.—When we refer hereafter to a nominal rate per annum, we shall write "per cent." in full, using for actual rates per period the symbol % or the decimal. The letters a, s, q, or m, will stand for "payable annually," "semi-annually," "quarterly," or "monthly."

132.—The following phrases need interpretation into more exact language:

(a) "Six per cent. per annum, payable annually," means what it says: six per cent. per term, the term being a year.

(b) "Six per cent. per annum, payable semi-annually," means three per cent. each half year; which is more than six per cent. per year.

(*c*) "Six per cent. per annum, payable quarterly," means one-and-one-half per cent. per term of three months.

(*d*) "Six per cent. per annum, payable monthly," means one-half per cent. per month.

133.—In cases (*b*), (*c*) and (*d*), the "6" is fictitious. The ratios which must be used are 1.03, 1.015 and 1.005, not 1.06 at all. "Six per cent." is known as the nominal rate, but the effectiv rate for the entire year is different.

Taking up the above four cases:

(*a*) Here the nominal and the effectiv rate are identical; .06.

(*b*) Here the effectiv rate is .03 per half year; for the year .0609.

(*c*) Here the effectiv rate is .015 per quarter; for the year .06136355.

(*d*) Here the effectiv rate is .005 per month; for the year .06167781.

134.—Thus the words "six per cent. per annum" have four different meanings, according to the qualifying phrase used, or understood. Let j represent the nominal rate "per annum," i being the rate per term, and k the effectiv rate per year.

Then in (*a*), where $r = 1.06$ and $t = 1$,

$$1 + k = 1 + j = 1.06$$

In (*b*), where $r = 1.03$, and $t = 2$,

$$1 + k = r^2 = (1 + \tfrac{1}{2}j)^2 = (1.03)^2 = 1.0609$$

In (*c*), where $r = 1.015$ and $t = 4$,

$$1 + k = r^4 = (1 + \tfrac{1}{4}j)^4 = (1.015)^4 = 1.06136355 \,||$$

In (*d*), where $r = 1.005$ and $t = 12$,

$$1 + k = r^{12} = (1 + \tfrac{1}{12}j)^{12} = 1.005)^{12} = 1.06167781 \,||$$

These values may be ascertained by logarithms or by arithmetic.

135.—Case (*b*) furnishes an arithmetical solution which is very convenient. Expanding $(1+j/2)^2$ by the binomial theorem we have $1 + j + j^2/4$. To the nominal rate the quarter of its square is to be added to give the effectiv rate if compounded

at half periods. Thus at 6% for j, $.06^2 = .0036$, $.0036/4 = .0009$; $.06 + .0009 = .0609$. At 8%, $.08^2 = .0064$; $.0064/4 = .0016$. $k = .0816$.

136.—The rate k being $= j + j^2/4$, we may factor this, making it $j(1 + j/4)$. $1 + j/4$ is thus a *multiplier*, reducing the nominal *rate* payable semi-annually to an effectiv annual rate. For six per cent. this multiplier would be 1.015, (.0609 / .06); for five, 1.0125; for four, 1.01; for 3%, 1.0075; for 2%, 1.005. The same reasoning applies to a nominal half-yearly rate, payable quarterly. If 3% is i for the half-year, 3 (1.0075) is j for the half year with quarterly payments, or 3.0225.

137.—But the annual rate given may be the effectiv rate (i) and the question be, what rate (j) will be equivalent for the case of more frequent payments, giving k as the nominal rate per annum for that frequency.

Case (a) is the same as before.

Case (b) $1 + j = (1 + i)^{\frac{1}{2}} = (1 + \frac{1}{2}k)$ For $i = 6\%$, $1 + j = (1.06)^{\frac{1}{2}} = 1.02956301$; and $k = 2j = .05912602$. That is, to produce 6% payable annually, we must invest at 5.912602% per annum, payable semi-annually, or 2.956301% per period of six months.

(c) $1 + j = (1 + i)^{\frac{1}{4}} = (1 + \frac{1}{4}k)$ For $i = 6\%$, $1 + k = 1.05869538$, payable quarterly.

(d) $1 + j = (1 + i)^{\frac{1}{12}} = (1 + \frac{1}{12}k)$ For $i = 6\%$; $k = .058269$, payable monthly.

138.—In annuity calculations the period or interval between cash payments is to be considerd as well as the frequency of compounding the interest. Here, also, the terms are reduced to the "per annum" standard. An annuity of $50 per half year is usually spoken of as an annuity of $100, payable semi-annually. What the actual value of the yearly revenue is, depends upon the rate of interest assumed in the problem.

139.—If $\frac{1}{2}a$ represents the instalment or "rent" of the annuity for each half-year, and i the rate of interest for the half-year, the equivalent of these two cash payments for the

year will be $\frac{1}{2}a + \frac{1}{2}a\,(1 + i) = a + \frac{1}{2}ai = a\,(1 + \frac{1}{2}i)$. If j is the nominal rate per annum or $2i$, then the annual effectiv payment is $a\,(1 + j/4)$ and $1 + j/4$ is a multiplier for transforming a yearly annuity into a half-yearly one. This is the same multiplier which was alredy found to transform a yearly nominal rate of interest, compounded semi-annually into its corresponding effectiv rate. This multiplier, $1 + j/4$, will be found important in practis. It may be called the co-efficient of double frequency, or $C^{(2)}$. The $^{(2)}$ represents the ratio of the frequency of compounding to that of payment.

140.—If the rate of interest is 3% per half year (6 per cent., s) and the annuity payment \$1 per annum, to find the amount of the annuity for four years, we may reduce the interest to the annual standard, the cash being alredy there.

The annual equivalent of the rate is .0609 (6×1.015).
Twice the logarithm of 1.03 .012837224705
is log. 1.0609 = .025674449410.
The first step is to find the amount, for which purpose the logarithm is multiplied by 4, .1026977976400.
This is also 8 times the logarithm of 1.03, so that we gained nothing by squaring 1.03. From either view the amount is 1.26677008 and the compound interest is .26677008. This is next to be divided by the rate of interest, which is not .03, nor .06, but .0609.

```
.0609).26677008(4.3804601, amount of annuity.
       2436
       ----
        2317
        1827
        ----
         4900
         4872
         ----
           2808
           2436
           ----
            372
            365
            ---
              7
```

141.—We may test this result as follows:

End of first year; cash		1.00000
" third half year;	interest .03 on 1.00	.03
" second year;	" .03 on 1.03	.0309
" " "	cash	1.00000
Total		2.0609
End of fifth half year;	interest .03 on 2.061	.06183
" third year;	interest .03 on 2.123	.06369
" " "	cash	1.
Total		3.18642
End of seventh half year;	interest .03 on 3.186	.09558
" fourth year;	interest .03 on 3.282	.09846
" " "	cash	1.
Total		4.38046

142.—We may simplify this method a little further. Had we made the instalment 50 cents each half year, the compound interest would have been half as much, or .13338504. This would have been divided by .03, giving 4.446168. It would have been the same had we divided the compound interest of \$1 by .06. But we did divide it by .0609, which is $.06 \times 1.015$, the latter being the coefficient of double frequency. We might, therefore, have divided the amount of the annuity when payable semi-annually by the $C^{(2)}$

$$4.446168 / 1.015 = 4.38046$$

143.—Therefore, an annuity payable annually is transformd as to its amount into one payable half-yearly by multiplying it by the $C^{(2)}$.

144.—The present worth of the annuity is subject to the same law; when the annual payment is divided into two equal sums its present worth is increast in the ratio of $1 + j/4$ or $1 + i/2$. In the case given above

the logarithm	.102697797 6400
would have been changed to its cologarithm	$\bar{1}$.8973022023600
the number of which would be the present worth	.78940923 4
The compound Discount would be	.210590766
and the rate or divisor as before	.0609
giving the present worth of the annuity as	3.4579764 5

145.—The correctness of this may be demonstrated as follows:

Amount invested in annuity............	3.45797645
Half-year's interest on 3.457976+	.10373929
	3.56171574
Half-year's interest on 3.561716—	.10685147
	3.66856721
Annual instalment....................	1.00000000
	2.66856721
Half-year's interest on 2.668567+	.08005702
	2.74862423
Half-year's interest on 2.748624+	.08245873
	2.83108296
Annual instalment....................	1.00000000
	1.83108296
Half-year's interest on 1.831083—	.05493249
	1.88601545
Half-year's interest on 1.886015+	.05658046
	1.94259591
Annual instalment....................	1.00000000
	.94259591
Half-year's interest on .942596—	.02827788
	.97087379
Half-year's interest on .9708738+	.02912621
	1.00000000
Last instalment......................	1.00000000

146.—Had the payments been half-yearly, each being 50 cents, the compound discount would
have been........................ .105295383
and we should have divided by .03,
giving........................3.5098461
Dividing by the $C^{(2)}$ 1.015, we should
again have the value............3.457976+

147.—The conclusion is that there are two ways of calculating the amount or present worth of an annuity where the interest compounds with twice the frequency of the cash payments.

(1) Procede as if both were at the greater interval, taking care to use the effectiv rate of interest in dividing.

(2) Procede as if both were at the smaller interval, the instalment being half as much and divide the result by the $C^{(2)}$.

148.—Where the interest-period is greater than the payment-period, or the payments are made twice as frequently as the interest is compounded, the solution is less easy because it depends on evolution.

149.—Half of the instalment is paid when only half the interest-period has elapst. It may be considered as earning interest for the other half-period, but the rate must be taken effectivly. Thus, if the interest for a period is .03, the ratio for the half-period is the square root of 1.03, or 1.01488916.

The half-instalment paid at the half-period becomes at the end of the full period...... .50 $\times$ 1.01488916 = .50744458
The other half-instalment is only .... .50

and the total of both is.................. 1.00744458

This is the effectiv instalment, insted of the nominal instalment, $1. It is a coefficient of frequency as to payments, and may be represented by $C^{(\frac{1}{2})}$, meaning that the interest is compounded only half as often as a payment is made.

150.—If the period of compounding is the half year at .03 per period (a nominal rate per cent. of .06), the effectiv rate per quarter is 1.01488916 and the $C^{(\frac{1}{2})}$ is 1.00744458, being half the square root of 1 + half the nominal rate. If the nominal rate is 3.8 per cent. (s), take first the logarithm of

1.019 .... .008 174 184 006,4
and divide by 2............ .004 087 092 003,2
The number corresponding to this
is.............................. 1.009 455 299
halving the decimal part........ 1.004 727 649

is the $C^{(\frac{1}{2})}$ for a rate of 3.8 per annum (s), payments quarterly, or $1 ($q$).

151.—The coefficient of frequency ($C^{(\frac{1}{2})}$) has to be computed at the commencement of each problem by the above method.

152.—The computation of the amount, or of the present worth, as the case may be, then goes on just as if the payments took place at the same times as the compoundings. When completed, the result is multiplied by $C^{(\frac{1}{2})}$.

153.—When the interest-period is semi-annual and the instalments are paid quarterly, it is better to ignore the "per

annum" rate and treat of the periods (half years) and half periods (quarters), after the commencement.

154.—An annuity of $2 per annum, payable quarterly, interest to be compounded semi-annually, for 2 years at $3\frac{8}{10}$ per cent. per annum, would be stated as an annuity of $1 per period, payable by half-periods interest at $1\frac{9}{10}$% per period, and continuing for four periods. The present worth of this annuity, omitting the condition "payable by half-periods," would be 3.81698703, which × 1.00472765 is 3.8350324, the present worth when the annuity is paid at the quarters or half-periods. Tested as follows:

Present worth	3.8350324
Interest at .019	.0728656
	3.9078980
First and Second Instalments, with interest on the first	1.0047276+
	2.9031704
Interest at .019	.0551602
	2.9583306
Third and Fourth Instalments, as before	1.0047277
	1.9536029
Interest at .019	.0371185
	1.9907214
Fifth and Sixth Instalments	1.0047276
	.9859938
Interest at .019	.0187339
	1.0047277
Seventh and Eighth Instalments	1.0047277

The $C^{(\frac{1}{2})}$ being almost exactly 1.00472765, it is taken alternately as 1.0047276 and 1.0047277.

155.—Values of $C^{(\frac{1}{2})}$ for all ordinary rates are found by taking half the decimal part of the figures under "Square Root" in Table VI. of the Text Book of the Accountancy of Investment, Part III, the "1" remaining where it is.

156.—To find the amount or the present worth of an annuity where half of each instalment is collected midway of the period, procede as if the entire instalment were collected at the end and then multiply the result by $C^{(\frac{1}{2})}$, being $1 + \frac{1}{2}(\sqrt{1+i} - 1)$.

157.—In some theoretical computations interest is conceived as compounding momently or continuously. Interest at 6% per annum, when compounded momently, gives an equivalent effectiv rate of .061837. This is obtained by multiplying the rate .06 by the constant quantity .4342944819 (or as many figures as required); considering this as logarithm, its number will be the *ratio* sought, 1.061836546539. If .06 is the effectiv rate and it is desired to find the nominal rate, multiply the logarithm of the ratio (L) by the constant quantity 2.302585092994, or so much as required and the result will be the nominal rate.

log. 1.06 = .0253058652648. This $\times$ 2.30585092994 = .0583689+. These constants depend on the Naperian logarithms.

FRACTIONAL PERIODS.

158.—We have hitherto treated only of entire periods, but it is quite usual that the number of periods should be a mixt number, sometimes a fraction only.

159.—A det is due in one year from now, at six per cent. annually; but the dettor has the privilege of paying at the half year; what interest should he then pay? There are two answers to this question, depending on whether it is to be considerd legally or equitably — by simple interest or by compound interest.

160.—Legally, the rate is .03 per half year, the law not recognizing the justice of compound interest. Equitably, that is not the true proportion in which the interest should be divided. The creditor gets, not six per cent. annually, but six per cent. semi-annually, which we have seen to be more profitable.

161.—The compound interest for a half term is at the rate of .02956301 only, not .03. Compound interest for several periods is greater than simple interest; conversely, for part of a period the compound interest is the lesser.

162.—If the det spoken of is $1,000,000 and is discharged at midyear by a payment of $1,030,000, the creditor has the

use for six months of $30,000, at *some* rate from which the dettor has no benefit, besides the use of the $1,000,000 to which he is entitled.

163.—If interest were not a constant force, but a periodical incident, there would be no such thing as interest between the periodical dates; one would have to pay a full period or nothing.

164.—The result of this inconsistency is that, conventionally, when interest is calculated on a certain number of terms and a fraction of a term, the interest compounds for the integral terms, but remains simple during the fraction of a term.

165.—For four-and-a-half years on the conventional interest plan at six per cent. annually, the compound interest must be calculated for four years; amount........ 1.26247696
then this must be multiplied by 1.03 (the conventional ratio for the half year) producing 1.30035127
This number is exactly midway, arithmetically, between the amount at four years.......... 1.26247696
and that at five years..................... 1.33822558

This plan of dividing the difference in proportion to the time elapst is generally used where the even periodical values can be obtaind from tables, especially in case of valuation of bonds, as will be shown hereafter.

166.—In scientific interest, the ½ forms part of the number of terms. The log. 1.06............ 025 305 865 264,8
being multiplied by 4.5 gives........... 113 876 389 191,6
the number for which is................ 1.29 979 957 070
This result might have been obtaind by multiplying the 4 year amount.......... 1.26 247 696
by the inconvenient number............. 1.02 956 301
which is the effectiv ratio.

167.—When annuities are to be sumd or valued, it is necessary to get the value for the entire terms first and then multiply by the effectiv rate for scientific interest; for conventional interest either multiply by the conventional rate, or "split the difference," according to time elapst. It is impossible to value or sum the annuity in one operation by a fractional multiplier, for the reason that these processes depend entirely on a uniform ratio.

168.—It is the universal custom in actual business to treat parts of terms by simple interest, not by compound; conventionally, not scientifically.

SINKING FUNDS.

169.—We have hitherto assumed the periodical instalment, or rent of an annuity, to be 1. When this is some other number, the amount or present worth of $1 is multiplied by that other number; that is, the amounts (or present worths) are directly proportionate to the rent. But sometimes we have given the amount or the present worth as a fixt sum and wish to find an instalment which will produce that amount or extinguish that present worth.

170.—We have seen that the amount of an annuity of $1 at 3% for 50 periods is $112.79687. If the amount were $1000 insted of $112.79687, it is evident that each instalment must be increast as many times as $112.79687 is containd in $1000. The quotient is 8.8655. Therefore, under the same conditions where $1 amounts to $112.79687, $8.8655 will amount to $1000. If the growth of the two annuities be compared it will be seen that at any point the one which is to accumulate to $1000 is 8.8655 times as large as the one which accumulates to $112.79687.

Instalments of $1	Instalments of $8.8655
1.0000	8.8655
.0300	.2660
1.	8.8655
2.0300	17.9970
.0609	.5399
1.	8.8655
3.0909	27.4024
.0927+	.8220
1.	8.8655
4.1836	37.0899
etc.	etc.

Therefore, to find the instalment which contributed each period, will amount to a given sum S, divide S by the amount of an annuity of $1.

171.—Where an annuity is so constructed that it shall accumulate to a certain amount at a certain time, it is called a sinking fund. Frequently the uniform periodical contribution is itself calld the sinking fund, and is found in the foregoing manner.

172.—Where the present worth is the quantity given, the process of finding the uniform contribution which will gradually extinguish or amortize that present worth by the aid of compound interest is similarly performd. The fixt quantity is the present worth of an annuity of x dollars ; the given present worth divided by P, the present worth of $1 gives the instalment, x, necessary to amortize it.

173.—It is required to find what annual payment will clear off $1000 in 50 periods, allowing .03 interest. We have alredy calculated that a payment of $1 per period will pay off $25.729764, with interest. $1000 is 38.8655 times $25.729764 ; therefore the contribution must be $38.8655 per period, which will, by forming a schedule, be found to amortize the $1000.

174.—As a provision for liquidating indettedness, or for replacing vanishing assets, sinking fund and amortization are two different applications of the same principle. Formerly, the terms were used interchangeably, but more recently they are distinguisht as follows:

> 175.—The sinking fund permits the det to stand till maturity, but in the meantime provides a fund which at maturity pays off the entire det, the Interest on the original sum being paid separately.
>
> 176.—The amortization plan accumulates nothing, but *gradually* reduces the det, applying to this reduction all the excess of the contribution over the Interest.

177.—The two operations which we have performd show that the sums necessary to be set aside for a det of $1000 during 50 periods at .03 are,

Sinking Fund..............	$ 8.8655
Amortization	38.8655
The difference is the........	$30

per period, which on the sinking fund plan is required to pay the current interest, so that actually the two methods of contribution come to the same thing.

178.—The number of terms necessary for a certain contribution per period to amount to a certain principal may be found, but first the amount of a single dollar must be found.

The amount of the annuity is I/i, the total compound interest divided by the rate of interest. Multiplying that amount by the rate gives, therefore, the compound interest. Adding to this $1 we have the amount of a single dollar, s or $S \times i + 1$. We then proceed as shown in Art. 93.

Similarly the present worth of the annuity being D/i, $p = 1 - P \times i$, and t may be deduced therefrom.

179.—The *rate of interest* of an annuity cannot be ascertaind by any direct formula, as it involvs the solution of equations of higher degrees.

180.—A special method for finding the income-rate of securities by gradual approximation will be given hereafter. (Art. 231).

INTEREST-BEARING SECURITIES.

181.—A bond (which is the most usual form of interest-bearing security) is a complex promise to pay:

1. A certain sum of money at a future time; this is known as the principal, the par or the capital.

2. Certain smaller sums, proportionate to the principal, and at various earlier times. These are usually known as the "interest," but as they do not necessarily correspond to the true rate of interest, it will be better to speak of them as the coupons.

182.—These various sums are never worth their face or par until the stipulated times arrive, but are always at a discount. The principal is never worth its face until its maturity; the coupons are never worth their face until the maturity of each. Yet while both principal and coupons are at a discount, the aggregate may easily be worth more than the par, and it is the aggregate, principal and coupons which is the subject of the valuation.

183.—If the bond is sold at par, the coupon and the interest are equivalent. Take a five per cent. (s) bond for $10,000, due in 5 years, at par. Its value consists of

— EXAMPLE 1 —

1. The present worth of $1000 at 10 periods at .025	781.1984
2. The present worth of an annuity of $25 per period, 10 periods	218.8016
Aggregate	1000.0000

— EXAMPLE 2 —

But if the coupons were $30 each, the bond being "six per cent.," the principal would still be valued at..............	781.1984
while the coupons would be worth	262.5619
Aggregate	1043.7603

— EXAMPLE 3 —

If the bond were a "four per cent." bond, the coupons being $20 each, the valuations would be, principal........	781.1984
coupons	175.0413
Aggregate	956.2397

All the above calculations may be made by logarithms, commencing with the logarithm (L) of 1.025.

184.—From these computations we may draw the following inferences:

1. If the coupon rate is the same as the income rate, the bond is at par.

2. If the coupon rate is greater than the income rate, the bond is worth more than par.

3. If the coupon rate is less than the income rate, the bond is worth less than par.

185.—Rule I. Any bond may be valued so as to earn a given interest rate by adding together

1. Present worth of the principal;
2. Present worth of the annuity, consisting of all the coupons.

186.—Representing the coupon rate or the proportion which the coupon bears to the principal, by c and the value of the bond for $1 by V

$$V = r^{-t} + c\frac{1 - r^{-t}}{i}$$

r^t is the only quantity which requires logarithms for its computation, which always begins with L, the logarithm of r. tL is the logarithm of r^t and subtracted from zero is the logarithm of r^t. In the above example

$$L \text{ or } \log.(1+i) \text{ or } \log. 1.025 = \quad .010\ 723\ 865\ 391{,}8$$
$$\log. 1.025^{10} = tL = \quad .107\ 238\ 653\ 918$$
$$\log. 1025^{-10} = \bar{1}.892\ 761\ 346\ 082$$
$$\bar{1}.892\ 761\ 346\ 082\ ln \quad .781\ 198\ 401\ 727$$

Substituting the above value for r^{-t} will give the results in Examples 2 and 3.

187.—In the second and third case the correctness of the figures may be proved by forming a schedule of amortization, which, starting with the present value, will bring the value, up or down, to par at maturity.

SIX PER CENT. BOND, NET INCOME .025.

Coupons	Interest at .025	Amortization	1043.7603
30.	25.0940	3.9060	1039.8543
30.	25.9964	4.0036	1035.8507
30.	25.8962	4.1038	1031.7469
30.	25.7937	4.2063	1027.5406
30.	25.6885	4.3115	1023.2291
30.	25.5808	4.4192	1018.8099
30.	25.4702	4.5298	1014.2801
30.	25.3570	4.6430	1009.6371
30.	25.2410	4.7590	1004.8781
30.	25.1219	4.8781	1000.0000
300.	356.2397	43.7603	

FOUR PER CENT. BOND, NET INCOME .025.

Coupons	Interest at .025	Amortization	956.2397
20.	23.9060	3.9060	960.1457
20.	24.0036	4.0036	964.1493
20.	24.1038	4.1038	968.2531
20.	24 2063	4.2063	972.4594
20.	24.3115	4.3115	976.7709
20.	24.4192	4.4192	981.1901
20.	24.5298	4.5298	985.7199
20.	24 6430	4.6430	990.3629
20.	24.7590	4.7590	995.1219
20.	24.8781	4.8781	1000.0000
200.	243.7603	43.7603	

In the six per cent. example the amortization is subtracted; in the four per cent. example the amortization of discount (called also accumulation or accretion) is added. The figures in the two amortization colums are identical.

188.—This is the most natural method of valuation, and for one who only occasionally employs it, perhaps the safest. There are other methods which in practis are briefer.

189.—The excess over par in the second example (six per cent. coupons), $43.7603, is known as **premium.**

In the third example (four per cent. coupons), the value is less than par and the difference is known as **discount**, a word which has several meanings. When I have occasion to speak of both premiums and discounts I shall use the word **variance**; that is, variance from par.

190.—The difference between the coupon-rate, or cash-rate, and the interest-rate, or income-rate, is the sole cause of the variance. This difference will be called the **interest-difference.**

.025 being assumed as the interest-rate, and the coupon-rate .03 or .02, the interest-difference is .005.

191.—Where the coupon is .03, the .025 may be considered as interest on $1, and each .005 is a future benefit or extra profit, which should be paid for. Reduced to present values, these benefits are the present worth of an annuity of .005 per period.

192.—The present worth of an annuity of .005 for 10 periods is .0437603, or on $1000, $43.7603, the same variance as found by the previous process.

193.—**Rule II.** The variance is the present worth of an annuity of the interest-difference. When the coupon-rate is the greater, the variance is added to the par; when the coupon-rate is the less, the variance is subtracted from par.

194.—Representing the variance by Q, the second rule may be exprest as follows:

$$Q = (c - i)\,\frac{1 - r^{-t}}{i}$$

$$\text{or } \frac{c - i}{i}\,(1 - r^{-t})$$

195.—Multiplying both numerator and denominator by 200 will not alter the value of the fraction, hence it will be the same thing if we use the nominal annual rates. Insted of $\frac{.03 - .025}{.025}$ we may use $\frac{6-5}{5}$ which is easier.

In the above example, No. 2, the variance would be obtaind thus:

$$Q = \frac{6-5}{5} (1 - .781198401727$$

$$V = 1 + \frac{6-5}{5} (1 - .781198401727$$

$$= 1 + \frac{1}{5} (.218801598273)$$

$$= 1 + .043760319655 = 1.0437603 +$$

In the third example, where $c = 4$

$$Q = \frac{4-5}{5} (.2188016)$$

$$= - \frac{1}{5} (.2188016)$$

$$V = 1 - .0437603 = .9562397$$

The results may be carried to 11 or 12 decimals if desired.

196.—A third method (suggested by Mr. Arthur S. Little) is based upon the value of a perpetual bond. This, as there is no redemption, is merely a perpetual annuity, or perpetuity of the "coupon." The value of such a perpetuity is c/i. A six per cent. (s) bond to pay five per cent. (s) is $.03/.025 = 6/5 = 1.20$, and this value is perpetual, there being no redemption. But if it is known that the variance (.20) will vanish 10 years from now, the value of the bond is now lessend by the present worth of that variance.

197.—Rule III. The terminant value is the perpetuity value, minus the present worth of the perpetuity variance.

$$V = \frac{c}{i} - \left(\frac{c}{i} - 1\right) (r^{-t})$$

In Example 2, the perpetuity value is 6/5 =	1.20
The present worth of the vanishing quantity .20 is	.1562397
Remainder....	1.0437603

Here the first step is to obtain the perpetuity value by simple arithmetic. The variance is, of course, .20. Then r^{-t}=.781 198 401 727 is found as usual and multiplied by .20, giving .... .156239680+

In Example 3, the perpetuity is..........	.80
but the variance is still .20, and its present	
worth is..............................	.15623968
which is *added*, making................	.95623968
because minus a minus is plus.	

198.—**Multiplying down.** Whichever of these three methods has been employd for ascertaining the value of the bond at a certain date, if the successiv values for each period are expected to be required (and they usually are), it is preferable to find them by schedules of amortization rather than resort to independent logarithmic calculation for each. A thoro test of the correctness of all the intermediate values is the fact that the series reduces to par at maturity. In this test, insted of a formal schedule in colums, all the figures may be brought into a single colum, so that no marginal computation may be needed. The amount of each amortization is not exprest, but implied, in the following example of such a single-colum schedule:

A four per cent. (*s*) bond for 4 years to net three 96/100 (*s*). The *r* is 1.0198:

1.0198	*nl*	.008 515 007 631 5
×8		.068 120 061 052 0
Subtract from zero:................		$\bar{1}$.931 879 938 95
(A85 B05 C67 D93 E75 F27)		
Present worth of $1................		.854 830 361 69
Compound discount................		.145 169 638 31
Divide by .0198		7.331 799 914.75
Multiply by interest difference.....		.000 2
Premium..........................		.001 466 359 98
Value.......................		1.001 466 359 98
.0198	.01..........	.010 014 663 60
	.009..........	.009 013 197 24
	.0008.........	.000 801 173 09
		1.021 295 393 91
	Coupon	.02
		1.001 295 393 91

We save a number of figures by adding and subtracting at the same time, putting a circle round the coupon to indicate subtraction :

Resuming..................	1.001 295 393 91
.01..........	10 012 953 94
.009.........	9 011 658 55
	2 801 036 32
	1.001 121 042 72

The operation may be still further abridgd by amortizing the premium only, but subtracting the interest-difference only, not the entire coupon :

	.001 121 041 72	(1)
	11 210 43	
	10 089 38	
	2 896 83	
	943 239 36	(2)
	9 432 39	
	8 489 15	
	2 754 59	
	761 915 49	(3)
	7 619 15	
	6 857 24	
	2 609 53	
	577 001 41	(4)
	5 770 01	
	5 193 01	
	2 461 60	
	388 426 03	(5)
	3 884 26	
	3 495 83	
	2 310 74	
	196 116 86	(6)
	1 961 17	
	1 765 05	
	2 156 89	
Error in decimals......	03	
	.000 000 000 00	(7)

The 2 is two places further to the right than in the first procedure.

199.—**Computing Amortizations.** It may sometimes be advisable to find and verify at first the instalments of amortization, leaving this series of amounts to stand, not filling in the remaining colums of the schedule, until required. The obtaining of the amortization colum is remarkably easy, as shown in Art. 119.

200.—Starting with the premium or discount at t periods, as above explaind, it is next amortized to the extent of $\frac{c-i}{r^t}$ that is the present worth of the interest-difference. This multiplied by r gives the next amortization, $\frac{c-i}{r^{t-1}}$ and so on down.

201.—In the last example, the premium at 4 years was found to be............................. .001 466 359 98
The next amortization is simply the present worth at 8 periods of .0002. The present worth of \$1 has been found to be .854 830 361 69; this × .0002 = .000 170 966 072 238
Three figures may be dropped from this with safety, leaving the decimals as much extended as in the previous operation.

1.000...........	.000 170 966 072	(1)
.01	1 709 661	
.009...........	1 538 695	
.0008..........	136 773	
	174 351 210	(2)
	1 743 512	
	1 569 161	
	139 481	
	177 803 355	(3)
	1 778 034	
	1 600 230	
	142 243	
	181 323 862	(4)
	1 813 239	
	1 631 915	
	145 059	
	184 914 075	(5)
	1 849 141	
	1 664 227	
	147 931	
	188 575 374	(6)
	1 885 754	
	1 697 178	
	150 860	
	192 309 166	(7)
	1 923 092	
	1 730 782	
	153 847	
	.000 196 116 887	(8)

These are the eight instalments of amortization, which, added together, should equal the total premium.

$$\begin{array}{r} .000\ 170\ 966\ 07 \\ 174\ 351\ 20 \\ 177\ 803\ 35 \\ 181\ 323\ 86 \\ 184\ 914\ 07 \\ 188\ 575\ 37 \\ 192\ 309\ 17 \\ 196\ 116\ 89 \\ \hline .001\ 466\ 359\ 98 \end{array}$$

But if this test were not available, the last amortization would have to be multiplied up by 1.0198, which should produce .000200.

$$\begin{array}{r} .000\ 196\ 116\ 887 \\ 1\ 961\ 169 \\ 1\ 765\ 052 \\ 156\ 893 \\ \hline .000\ 200\ 000\ 001 \end{array}$$

202.—Small discrepancies in the last figure are to be expected and disregarded; therefore, the decimals should be carried beyond the figures which are to be utilized.

203.—Discounting. A series of values in reverse order, beginning from maturity, may be obtaind (without using logarithms) by division, the interest-ratio being the divisor. The entire amount to be receivd on the above bond is: principal $1, coupon .02, total $1.02. This should be divided by 1.0198.

$$1.02 \div 1.0198 = 1.000\ 196\ 116\ 89$$

which is the value 1 period before maturity. The coupon .02 must be added before the second discounting process.

$$\begin{array}{rr} 1.020\ 196\ 116\ 89 \div 1.0198 = & 1.000\ 388\ 426\ 03 \\ & .02 \\ \hline & 1.020\ 388\ 426\ 03 \end{array}$$

$$1.020\ 388\ 426\ 08 \div 1.0198 = 1.000\ 597\ 001\ 41$$

This is a laborious process, even if, insted of dividing, we multiply, by the reciprocal of 1.0198, .980584428.

	1.02
	.980 584 428
	.019 611 689
	1.000 196 117
+ coupon	.02
	1.020 196 117
	.980 584 428
	.019 611 689
	098 058
	88 253
	5 884
	98
	10
	7
	1.000 388 427

Table of Multiples

1.	.980 584 428
2.	1.961 168 857
3.	2.941 753 285
4.	3.922 337 713
5.	4.902 922 142
6.	5.883 506 570
7.	6.864 090 998
8.	7.844 675 427
9.	8.825 259 855

204.—**Intermediate Purchases.** It happens very often (perhaps in a majority of cases), that bonds are not purchast on the very day when the interest is payable. In the preceding examples it was supposed that exactly 8 or 10 periods would elapse from the purchase of the bond till its maturity; but the purchase may have been a month, or several months, or months and days, after the beginning of the period.

205.—We saw (Art. 167) that an annuity cannot be valued by the usual formula when the number of terms is a mixt number. We must derive it from the next regular term or interpolate it between the two nearest. It was also explaind that this interpolation may be done in two ways: one by simple proportion, conventionally, or by compound interest, scientifically. In the present state of knowledge the conventional or non-scientific method is establisht by usage, altho it works injustice to the buyer. The difference is usually not very large.

206.—Let us suppose that in the above Examples, in Art. 182 the purchase had been made $9\frac{1}{2}$ periods before maturity, that is, 4 years 9 months.

The value at 10 periods (Ex. 2) is...	1043.7603
the value at 9 periods is	1039.8543
the difference, or amortization, is....	3.9060
As half the time has elapst, we assume that half the amortization has taken effect....	1.9530
and this we subtract from the 10-period value......	1043.7603
making.....	1041.8073

which is exactly half way between the 9-period and the 10-period values. Besides this, however, the purchaser must pay one-half of the current coupon, or $15.00, as "accrued interest," the entire cost being 1056.8073. This is called the flat price, and formerly this was the form in which securities were quoted at the Exchanges; now, however, the quotations are understood as so much "and interest," meaning that the accrued interest is made a separate item in the bill.

207.—Had the 10-year value been multiplied by 1.0125 (half the interest rate) the same flat value would have been obtaind. $1043.7603 \times 1.0125 = 1056.8073$

208.—To apply the scientific plan, the 1043.7603 would have been multiplied by 1.01242284, giving 1056.7267

insted of........................	1056.8073
a difference of..............	.0806

in favor of the purchaser, but he could not claim it under the law and the customs of the market.

209.—As the usual period is divided into six months, and as the odd days are considerd as thirtieths of a month, the amortization for each day is 1/180 of that for a half year. If 2 months, 17 days had past after the interest-date, then 1.9530 must be multiplied by 77/180, giving .8359 as the proportionate amortization for 77 days.

210.—Thus the value of a bond at any date from its issue to its maturity, at some given rate of interest, may be calculated by first valuing it at at two consecutiv interest dates, according to the rules given, and then "splitting the difference" by dividing it into 180ths.

211.—Even if the unit of time employed in the coupon-payments and the interest-compoundings be different, the rules given in the section entitled "The Unit of Time" will enable these to be allowed for, if Rule I (Art. 185) is used for valuing.

212.— **Intermediate Balances.** — When the regular interest-periods do not coincide with the date of the balance sheet, it becomes necessary to adjust the valuations for that purpose in the manner just described for purchases at odd times.

213.—If the interest-dates are May 1 and November 1, and the dates for balancing are January 1 and July 1, the bond must, on January 1, have been amortized to the extent of one-third of that from November to May, conventionally.

214.—A six per cent. (s) bond for $1000, to yield five per cent. (s) due Nov. 1, 1925, is worth on

November 1, 1920	1043.7603
on May 1, 1921	1039.8543
the amortization for 6 months is, therefore	3.9060
For 2 months it is one-third	1.3020
and this subtracted from	1043.7603
leave the value on Jan. 1, 1921	1042.4583
Applying the same method between May 1 and Nov. 1, 1921	1039.8543
	1035.8507
3)	4.0036
	1.3345
	1039.8543
we have the balance, July 1, 1921	1038.5198
If we take the January value	1042.4583
and multiply down; × .025	26.0615
	1068.5198
— Coupon	30.
we also get the July result	1038.5198

Thus we have the choice of interpolating each balance-value, or having obtaind one, of multiplying down to maturity,

which can be done on the conventional, but not on the scientific plan. The resulting schedule would be as follows:

Date	Collected	Interest at .025	Amortization	Value
Jan. 1, 1921	Value at	5 per cent.	basis	1042.4583
July 1, "	30.00	26.0615	3.9385	1038.5198
Jan. 1, 1922....	30.00	25.9630	4.0370	1034.4828
July 1, "	30.00	25 8621	4.1379	1030.3449
Jan. 1, 1923....	30.00	25.7586	4.2414	1026.1035
July 1, "	30.00	25.6526	4.3474	1021.7561
Jan. 1, 1924....	30.00	25.5439	4.4561	1017.3000
July 1, "	30.00	25.4325	4.5675	1012.7325
Jan. 1, 1925....	30.00	25.3183	4.6817	1008.0508
July 1, "	30.00	25.2012	4.7988	1003.2520
Nov. 1, "	20.00	16.7480	3.2520	1000.0000

The last period is of only 4 months, July 1 to Nov. 1, ⅔ of the half year. The cash collected is, therefore, considerd as only $20, ⅔ of $30 ; each previous coupon had included ⅓ of the following half year, and this must now be squared up. In the colum "Interest at .025" the procedure is peculiar.

The 16,7480 is composed of two parts :

1.	⅔ of .025 on the $1000 par..........	$16.6667
2.	.025 *in full* on the $3.2520..........	.0813
		$16.7480

215.—To explain why interest *in full* for the half-year is reckond on the premium, go back to the normal schedule in Art. 187, and it will be seen that the premium on May 1 was 4.8781. Now, on the conventional plan, based on simple interest, this 4.8781 should not vary during the period ; therefore the interest ought to be :

⅔ of .025 of 4 8781 = .0813

which is the same as .025 of ⅔ of 4.8781 (3.520) = .0813

Hence in the last or broken period the variance from par must be treated as having earnd interest during the entire period, while the par itself has only earnd interest for the actual time, as four months.

216.—It will also be notist that 3.2520 is ⅔ of 4.8781, so that if we had calculated 4.8781 by discounting, it would have been sufficient confirmation of the preceding values to take ⅔ of 4.8781 and compare it with 3.2520.

217.—It must be rememberd that the periods introduced for balancing purposes are artificial, and that, strictly speaking, amortization takes place only at the dates when interest becomes due. The charging of part of the coupon, tho not yet collected, is fictitious, but in each period until the last, this borrowing is compensated for by a fresh loan.

218.—**Short periods, terminal or initial.**—It happens sometimes (altho it should be avoided) that the bond does not mature at the end of an interest period, but at some previous date. This gives rise to a fractional period, not an artificial one like those establisht for balancing purposes (Art. 212), but an actual one, which must be taken into account in the valuation.

219.—We will take the case of a six per cent. (s) bond for \$1000, issued Jan. 1, 1921, and payable Nov. 1, 1925, interest payments January and July 1, and valued to pay five per cent. (s). There are 9 full periods and a short period of 4 months, or ⅔ of a period. The coupon for this short period would be \$20 insted of \$30, as in such cases the last coupon is always proportional to its time. The interest ratio is also reduced for that period to $1 + \frac{2}{3}$ of .025, or 1.016⅔ by the conventional plan.

220.—Using the first method of evaluation, the following are the components of the value:

An annuity of 9 terms of \$30 each at .025....		\$239.1260
The present worth of \$1020 at .025 for 9 terms and .016⅔ for 1 term.		
\$1 for 9 terms...........	.8007284	
Divided by 1.016⅔......	.7876017	
Multiply by 1020.........................		803.3537
		1042.4797
It will be observd that this differs slightly from the value obtaind in Art. 209 for the same length of time, but with interest May and November...........................		1042.4583

The \$1020 referd to is composed of the principal and the last or partial coupon.

221.—To divide by a number like 1.016⅔, it is easier first to multiply both divisor and dividend by 3, converting each into a whole number.

$$1.0116\tfrac{2}{3} \times 3 = 3.05$$
$$.8007284 \times 3 = 2.40218508$$
$$2.40218508 \div 3.05 = .78760167$$

222.—To illustrate the case of an initial short period, suppose that the above bond had been issued on October 1, 1920, 3 months earlier than has been assumed; issued Oct. 1, 1920, interest January and July, principal maturing Nov. 1, 1925. There is then a preliminary coupon for 3 months, $15, to be discounted at 1.0125; 9 coupons of $30 each forming an annuity; one coupon of $20 and the principal, $1000, discounted for 9 periods at 1.025, and one period at 1.016⅔. The value on Jan. 1, 1921, obtaind as before, is 1042.4797. The simplest way will probably be to add to this the initial coupon $20, and discount back the entire 1062.4797 by dividing by 1.015, giving 1046.7780.

223.—We may have two other complications: the bond may be purchast within one of the odd periods; the balancing period may be at still another date.

224.—If the above bond were bought on Dec. 1, 1920, the price would be between 1046.7780, the October value, and 1042.4797, the January value, a three months' interval. The difference is 4.2983; and either ⅓ of this (1.4328) may be added to the January value or ⅔ (2.8655) subtracted from the initial value.

$$1042.4797 + 1.4328 = 1043.9125$$
$$1046.7780 - 2.8655 = 1043.9125$$

225.—The adjustment of values to balancing dates presents no special difficulty, being performd by simple proportion.

226.—**Cash payments on principal.**—Bonds at the same rates of coupon and of interest, tho at different dates of maturity, may be combined into one schedule. This may be done even if the interest-dates are different, but it is practically better in that case to keep the schedules distinct.

227.—We must commence with an aggregate value, made up of the separate values of the groups of bonds maturing on the same day.

228.—$2000 six per cent. (*s*) bonds, maturing as follows: $1000 on Nov. 1, 1923, $1000 on Nov. 1, 1925; interest .025, interest payments May and November. Required, value on Nov. 1, 1920.

The value of the first bond is...............	1018.8099
The value of the second.....................	1043.7603
Aggregate value....................	2062.5702

This value, multiplied down in the usual manner, gives the following schedule. At the date of the maturity of bond No. 1, the cash colum must contain not only the coupon $60, but the $1000 payable on principal.

Date	Cash Collections	Interest at .025	Payments on Principal	Investment Value
1920 Nov. 1				2062.5702
1921 May 1	60.00	51.5643	8.4357	2054.1345
" Nov. 1	60.00	51.3534	8.6466	2045.4879
1922 May 1	60.00	51.1372	8.8628	2036.6251
" Nov. 1	1060.00	50.9156	1009.0844	1027.5407
1923 May 1	30.00	25.6885	4.3116	1023.2291
" Nov. 1	30.00	25.5808	4.4192	1018.8099
	etc.	etc.	etc.	etc.

229.—The remainder of the schedule continues as in Article 187.

230.—For intermediate balances (Article 214), the interest requires adjustment at the date of partial payment. We now assume that the above group of bonds is to be valued on January 1 and July 1 of each year. The values of the bonds under consideration for January 1, 1921, would be, by interpolation 1017.3000
and......... .... 1042.4583
Aggregate.................................. 2059.7583

Dates	Cash Collections	Interest at .025	Payments on Principal	Investment Value
1921 Jan. 1				2059.7583
" July 1	60.00	51.4940	8.5060	2051.2523
1922 Jan. 1	60.00	51.2813	8.7187	2042.5336
" July 1	60.00	51.0633	8.9367	2033.5969
1923 Jan. 1	1050.00	42.5066	1007.4934	1026.1035
" July 1	30.00	25.6526	4.3474	1021.7561
	etc.	etc.	etc.	etc.

231.—The entries under January 1, 1923, are peculiar. The $1000 paid off was only in possession for 4 months, ⅔ of a period; therefore, $20 is the appropriate sum to be considerd as paid with it, and if it was kept in a separate account, that is all which would be allocated to it. The other $1000 is on interest during the full period and $30 is charged to it.

Cash entries:

For bond No. 1 par..................	1000
Gross interest thereon, .03, 4 months..	20
Gross interest on No. 2, .03, 6 months..	30
	1050

Interest entries:

Bond No. 1, ⅔ of period at .025.......	16.6667
Bond No. 2, full period, (Art. 210), at .025 on 1033.5969................	25.8399
	42.5066

Applied on principal:

1050.00—42.5066=................	1007.4934

232.—While this procedure may be applied where the successiv partial payments on principal are of the most irregular amounts and intervals, their chief utility is in what are known as serial bonds, where regular payments of principal are made, usually annually. Each bond or group of bonds, as it is dropt from the total, carries with it the appropriate cash and interest entries exactly as exemplified above.

TO FIND THE INCOME RATE.

233.—When the cash-rate, the time and the price of the bond are known, it is very desirable to know what is the income-rate, for, of course, every one wishes to get the highest income, security being equal.

234.—There is no positiv, direct method of doing this beyond three periods, as an equation of a higher degree is not directly soluble. There are only methods of approximation and trial.

235.—When printed tables are accessible it is easy to make a rough approximation by observing between what values the given price lies. The smaller the interval between the rates of the table, the closer is the approximation, and an additional decimal may be obtaind by proportion. With an extended table at close intervals, the result is sufficiently accurate for commercial purposes.

236.—There is a method devised by the author which will produce even greater accuracy, up to 12 places, at the expense of considerable labor. It appears in the later editions of his "Text-Book of the Accountancy of Investment," from which it is here quoted.

TO FIND THE INCOME RATE.

1.—Given a bond on which there is a premium or discount Q, cash-rate c payable in n periods, what is the income-rate, i?

2.—Every premium or discount is the present worth of an annuity of n terms, each instalment of which is the difference of rates; or it is the difference of rates $\times$ such an annuity of \$1 (Art. 193). Writing P for the present worth of an annuity of \$1 (Art. 108): $Q = P \times (c - i)$. If instead of the premium on \$1 we use that on \$100, we have $100\,Q = P \times (100c - 100i)$. It will not affect the value of the right-hand side if we halve one factor and double the other. $100\,Q = \frac{1}{2}\,P \times (200c - 200i)$. $200c$ is the rate *per cent. per annum* as conventionally termed. Thus we pay 4 per cent. per annum, meaning .02 per period. So also of $200i$ for i.

3.—It is evident that if we divide 100Q, the premium on \$100, by $\frac{1}{2}$P (which we will hereafter call the trial-divisor), we shall find the difference of rates. But as the annuity depends on the unknown rate, this does not help us at all.

4.—Let us *assume* the rate of income per annum to be any rate whatever, and calculate the trial-divisor at that rate. Then there is this property: If the assumed rate is too large, the quotient or difference of rates will be too small, and yet will be nearer the truth, and vice versa. From this approximate difference of rates we derive a new rate and proceed with this as a trial-rate.

The result of this trial will give a new rate still nearer and so on. We may slightly modify any rate to make it more easy to work. If we select our first trial-rate near the true rate, fewer successiv approximations will be necessary.

5.—Fortunately for our purpose, any table of bond values will readily give the trial-divisor, by taking the difference between the values at the same income-rate of two successiv \$100 bonds, say a 3% and a 4%, a 5% and a 6%, always 1% apart.

6.—For example, a 6% bond for \$100 (semi-annual) for 50 years is sold at 133.00, what is the income rate?

7.—With so large a premium as 33, the income-rate is evidently much less than 6. Let us assume 4. Then from any bond table we find on the 4% line the value of a 5% bond

to be..........	121.55
and that of a 4% bond to be................	100.00
The first trial-divisor is therefore............	21.55

$33.00 \div 21.55 = 1.531$, the difference of rates. $6 - 1.531 = 4.469$, the new trial rate. Taking 4.45 as more convenient, the new trial-divisor is 19.98. $33.00 \div 19.98 = 1.651$. $6 - 1.651 = 4.349$. We find that 20.315 is the trial-divisor for 4.35. $33.00 \div 20.315 = 1.6244$. $6 - 1.6244 = 4.3756$. Next using 4.37, trial-divisor 20.25: $33.00 \div 20.25 = 4.37$, almost exactly, so that 4.37 has reproduced itself. The value of the bond at 4.37, as computed by logarithms, is 133.0069, an error of less than one cent.

8.—It will be noticed in the foregoing example that the results always swing to the opposit side of the true rate; that is, if the trial-rate is too large the next rate is too small, and the true rate is between them. The successiv rates were 4...4.469...4.349...4.3756...4.37. 4.37 lies between any pair of these. This is always the case with bonds above par. With bonds below par it is different. The true rate always lies *beyond* the approximation.

9.—As an example of a bond below, take a 3% bond payable in 25 years. If purchased at 88.25, what is the income-rate? The following may be the steps, the dividend being always 11.75, the discount.

Trial-rates........	3.70	3.725	3.7265
Trial-Divisor	16.2190	16.175	16.17245
Result	3.7244	3.7264	3.7265

As 3.7265 reproduces itself, it must be correct to the 4th decimal; and the value of a 3% bond for 25 years to yield 3.7265 is found by logarithms to be 88.25015.

237.—Since the publication of the above method some simplifications have been suggested, referring particularly to the employment of Sprague's **Extended Bond Tables.** Insted of obtaining a divisor as shown in par. 5, we may first multiply Q itself by the interest-difference and then use the entire variance at the trial-rate (Q′) as the divisor, giving the same result. Thus in par. 7, where the first assumed rate is 4% and the interest difference 2, finding from the Table that the premium is........................ 430,983.52

	$33 \times 2 \div 43.098352$	$= 1.531$
insted of	$33 \div 21.55$	$= 1.531$

238.—The valuable suggestion has further been made by Mr. E. S. Thomas that by using the two nearest income rates and interpolating, five decimals may be obtaind at once. In the example where $Q = 33.00$, $200c = 6$, select 4.35 as $200i_1$ and 4.40 as $200i_2$. From the Tables we find opposit 4.35, 335,201.06 and opposit 4.40, 322,371.36, the interest-differences being 1.65 and 1.60.

4.35%	$33.00 \times 1.65 \div 33.520106$		$= 1.624398$
4.40%	$33.00 \times 1.60 \div 32.237176$		$= 1.637861$
	$4.35 + 1.624398$	$= 5.974398$	$= 6.- 025602$
	$4.40 + 1.637861$	$= 6.037861$	$= 6.+.037861$
	Difference in error		0.063463

239.—If the same operation be performd on a third rate, 4.45 :

4.45%	$33.00 \times 1.55 \div 30.974371$	$= 1.651333$
4.40%	$33.00 \times 1.60 \div 32.237176$	$= 1.637861$
	$4.45 + 1.651333$	$= 6.101333$
	$4.40 + 1.637861$	$= 6.037861$
	Difference	.063472

which differs so slightly from .063463 that further differencing may be neglected.

240.—By proportion, we may now ascertain at what rate the coupon will become 6. From 4.35 to 4.40, an extent of .05, it varies by .06347.

Therefore .06347 : .025602 : : .05 : .0201686

Add 4.35

Rate required 4.3701686

or safely 4.37017

241.—In the other example (par. 9), where Q = 11.75, it appears from the Tables, (p. 10), that the nearest rates are 3.70% and 3.75%; the corresponding values being 88.646703 and 87.900397 and the discounts 11.353297 and 12.099603.

3.70%	$- 11.75 \times .70 \div 11.353297 =$	$- .72445916$
3.75%	$- 11\ 75 \times .75 \div 12.099603 =$	$- .72832968$
3.80%	$- 11.75 \times .80 \div 12.837828 =$	$- .73221109$

	Diff.
$3.70 - .72445916 = 2.97554084$	
$3.75 - .72832968 = 3.02167032$	.04612948
$3.80 - .73221109 = 3.06778891$	04611859

$3 - 2.97554084 = .02445916$

.04612 : .02445916 : : .05 : .02650169

3.70

3.72650169

or safely 3.7265

242.—The process may be still further abridged by making the interest-difference the standard of comparison insted of the coupon, giving the same result. Thus in 238, insted of

$4.35 + 1.624398 = 5.974398 = 6. - .025602$

$4.40 + 1.637861 = 6.037861 = 6. + .037861$

.063463

might have been written

$1.65 - 1.624398 = \quad + .025602$

$1.60 - 1.637861 = \quad - .037861$

.063463

And in 241

$.70 - .72445916 = - .02445916$	
$.75 - .72832968 = + .02167032$	.04612948
$.80 - .73221109 = + .06778891$	.04611859

INTEREST FORMULAS.

$i =$ Rate of Interest, or the Interest on Unity for 1 period.

$r = 1 + i$, or the Ratio of Increase.

$t =$ Number of Periods of Time.

$r^{t} = (1 + i)^{t} =$ Amount $= s$.

$r^{t} = (1 + i)^{-t} = \dfrac{1}{(1 + i)^{t}} =$ Present Worth $= p$.

$r^{t} - 1 =$ Compound Interest $=$ I.

$1 - r^{t} =$ Compound Discount $=$ D.

Amount of Annuity $= 1 + r + r^{2} + r^{3} \ldots + r^{t-1} = \mathrm{I}/i = \mathrm{S}$.

Present Worth of Annuity $= 1 + r^{1} + r^{-2} + r^{3} \ldots r^{1-t} = \mathrm{D}/i = \mathrm{P}$.

$j =$ Nominal Rate Per Annum. Coefficient of Double Frequency $= \mathrm{C}^{(2)} = 1 + j/_{4}$. Coefficient of Half Frequency $= \mathrm{C}^{(½)} = 1 + ½ (\sqrt{1 + i} - 1)$

Sinking Fund $= 1/\mathrm{S} = i/\mathrm{I}$.

Amortization $= 1/\mathrm{P} = i/\mathrm{D} = i/\mathrm{I} + i$.

$c =$ Coupon Rate of Bond, or Cash Rate.

V $=$ Value of Bond at c to Earn $i = r^{-t} + c\,\mathrm{P}$.

or $= 1 + (c - i)\,\mathrm{P}$.

or $= \frac{c}{i} - (\frac{c}{i} - 1)\,\mathrm{P}$.